Warm Up

1. **Vocabulary** An equation or inequality that is always true is called a(n) __________ (identity, solution).

Solve each inequality.

2. $6x \leq 42$
3. $\frac{3k}{4} > \frac{5}{8}$
4. $2p + 3 < -25$
5. $5x - 3 - 7x \leq -9$

New Concepts

Sometimes an inequality will have a variable on both sides of the inequality sign. A solution to such an inequality is found by transforming the inequality so that the variable is isolated on one side of the inequality.

**Example 1** Solving Inequalities with Variables on Both Sides

Solve and graph each inequality.

**a.** $2x + 7 > -5x + 21$

**SOLUTION**

$2x + 7 > -5x + 21$

$7x + 7 > 21$ Add 5x to both sides.

$7x > 14$ Subtract 7 from both sides.

$x > 2$ Divide both sides by 7.

Graph the inequality on a number line.

**b.** $-\frac{5b}{8} + \frac{5}{16} \geq \frac{b}{8} - \frac{9}{16}$

**SOLUTION**

$-\frac{5b}{8} + \frac{5}{16} \geq \frac{b}{8} - \frac{9}{16}$

$-\frac{6b}{8} + \frac{5}{16} \geq -\frac{9}{16}$ Subtract $\frac{b}{8}$ from both sides.

$-\frac{6b}{8} \geq -\frac{14}{16}$ Subtract $\frac{5}{16}$ from both sides.

$b \leq \frac{7}{6}$ Multiply both sides by $-\frac{8}{6}$.

Graph the inequality on a number line.
Example 2  Simplifying Each Side Before Solving

Solve and graph the inequality.

\[2(x - 8) - 3x > 6 - 3(2x + 4)\]

**SOLUTION**

\[2(x - 8) - 3x > 6 - 3(2x + 4)\]

\[2x - 16 - 3x > 6 - 6x - 12\]  
Distributive Property

\[-x - 16 > -6x - 6\]  
Combine like terms.

\[5x - 16 > -6\]  
Add 6x to both sides.

\[5x > 10\]  
Add 16 to both sides.

\[x > 2\]  
Divide both sides by 5.

Graph the inequality on a number line.

Inequalities can be sometimes true, always true, or never true (false).

An inequality or equation that is always true is called an identity.

A **contradiction** is an inequality or an equation that is never true (false).

Math Language

The **solution set** is the set of values that makes an inequality true. If the inequality is a contradiction, then the solution set is empty, represented by \(\emptyset\).

Example 3  Solving Special Cases

Determine whether each inequality is sometimes true, always true, or never true (false). If it is sometimes true, identify the solution set.

a. \(3x + 4 - x > 2x + 7\)

**SOLUTION**

\[3x + 4 - x > 2x + 7\]

\[2x + 4 > 2x + 7\]  
Combine like terms.

\[-2x \quad -2x\]  
Subtraction Property of Inequality

\[4 > 7\]  
\(✗\)

The inequality is never true (false), so it is a contradiction.

b. \(3(2y - 6) \leq 6y + 4\)

**SOLUTION**

\[3(2y - 6) \leq 6y + 4\]

\[6y - 18 \leq 6y + 4\]  
Distributive Property

\[-6y \quad -6y\]  
Subtraction Property of Inequality

\[-18 \leq 4\]  
\(✓\)

The inequality is always true, so it is an identity.
Example 4  Application: Dog Breeds

The table shows the average number of American Kennel Club registrations (rounded to the nearest 100) for Bernese mountain dogs and rottweilers.

**American Kennel Club Registrations, 2002–2006**

<table>
<thead>
<tr>
<th>Breed</th>
<th>2006 Registrations</th>
<th>Average Yearly Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bernese Mountain Dogs</td>
<td>3700</td>
<td>300</td>
</tr>
<tr>
<td>Rottweilers</td>
<td>14,700</td>
<td>−1900</td>
</tr>
</tbody>
</table>

If the trend continues, in which year will the number of registered Bernese mountain dogs be equal to or exceed the number of registered rottweilers?

**SOLUTION**

Write an expression for the number of registered dogs for \( y \) years after 2006.

Bernese mountain dogs: \( 3700 + 300y \)

rottweilers: \( 14,700 - 1900y \)

\[
3700 + 300y \geq 14,700 - 1900y \\
3700 + 2200y \geq 14,700 \\
2200y \geq 11,000 \\
y \geq 5
\]

Add 1900\(y\) to both sides. Subtract 3700 from both sides. Divide both sides by 2200.

The solution \( y \geq 5 \) does not answer the question. The variable \( y \) represents the years after 2006.

2006 + 5 = 2011

If the trend continues, the number of Bernese mountain dogs registered will be equal to or exceed the number of rottweilers registered in 2011.

**Lesson Practice**

Solve and graph each inequality.

\[
a. \quad 4x - 8 > -2x + 4 \\
b. \quad \frac{3a}{5} + \frac{7}{10} \geq \frac{2a}{5} - \frac{9}{10} \\
c. \quad 4(x - 1) - 2x \leq 6 - 5(x + 2) \\
d. \quad x + 5 + 3x > 4x + 19 \\
e. \quad x + 5 > x - 3 \\
f. \quad \text{The table shows the average number of cell phone minutes used by Ara and Lexi.}
\]

<table>
<thead>
<tr>
<th>User</th>
<th>January Average</th>
<th>Average Change, January–May</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ara</td>
<td>4000</td>
<td>500</td>
</tr>
<tr>
<td>Lexi</td>
<td>12,000</td>
<td>−500</td>
</tr>
</tbody>
</table>

If the trend continues, in which month will Ara’s average minutes be equal to or greater than Lexi’s?
Factor completely.

1. \( w^2 - 13w + 36 \)
2. \( -q^2 + q + 42 \)
3. \( 30x^2 - 7xy - 2y^2 \)
4. \( x^2 - 11 + 6x - 44 \)

Solve.

5. \( |x - 3| = 14 \)
6. \( |x + 4| = 7.5 \)
7. \( -5 - \frac{n}{8} \geq -6 \)
8. \( 12 - 3d \leq -3 \)

Solve and graph. Then check the solution.

9. \( 6v + 5 > -2v - 3 \)
10. \( y + 4.5 < 10 \)

Write an equation for each of the lines described.

11. a line that passes through \((1, -2)\) and is perpendicular to \( y = 2x + 6 \)
12. a line that passes through \((6, 5)\) and is parallel to \( y = -x + 4 \)

13. **Multi-Step** To win a game, Alvaro needs to spin a black section or the number 10.

a. What is the probability of spinning a black section?
b. What is the probability of spinning a 10?
c. Are the events inclusive or mutually exclusive?
d. What is the probability of Alvaro hitting a black section or the number 10?

14. **Employment** The table below shows the number of people employed in the United States. Make a scatter plot of the data.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Employment (in millions)</td>
<td>79</td>
<td>86</td>
<td>99</td>
<td>107</td>
<td>119</td>
<td>125</td>
<td>137</td>
</tr>
</tbody>
</table>

15. **Verify** Show that 2 is a solution of the compound inequality \( x < 3 \) OR \( x > 6 \).

16. **Gardening** Debra is planting a square garden. The side length is \( 8 + \sqrt{8} \) inches. Write an expression to find the area of the garden, and then find the area.
17. Measurement To find the values of $x$ for which the triangle would have a perimeter of more than 81 units, solve the inequality $x + (x + 13) + (2x + 12) > 81$.

18. Write How do changes to the value of $b$ affect the graph of $y = \frac{a}{x - b} + c$?

19. Two number cubes are rolled and their values are added. One number cube is labeled 1–6. The other has two of each of the numbers 1, 2, and 3. The possible outcomes are displayed in the table.

What is the theoretical probability of each possible sum?

<table>
<thead>
<tr>
<th>Cube 1</th>
<th>Cube 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

20. Multiple Choice What is the horizontal asymptote for the rational expression $y = \frac{-7}{x - 2} + 4$?

A $y = -2$ B $y = 4$ C $y = -4$ D $y = 7$

21. Football A football is kicked into the air. In the expression $-5t^2 + 25t - 30$, $t$ represents the time when the ball is 30 feet in the air. Factor the expression completely.

22. Error Analysis Two students factor $9m^2x^3 + 81mx^3 + 126x^3$. Which student is correct? Explain the error.

<table>
<thead>
<tr>
<th>Student A</th>
<th>Student B</th>
</tr>
</thead>
</table>
| $9m^2x^3 + 81mx^3 + 126x^3$
| $= 9x^3(m^2 + 9m + 14)$
| $= 9x^3(m + 2)(m + 7)$ |
| $9m^2x^3 + 81mx^3 + 126x^3$
| The GCF is $9x^3$. Factor out $9x^3$.
| $m^2 + 9m + 14$
| $= (m + 2)(m + 7)$ |

23. Games Two number cubes are rolled and their values are added. Find the probability that the sum is less than or equal to 7.
**24. Geometry** Students made five tetrahedrons, a three-dimensional figure with four triangular faces, and labeled the faces 1–4. Use an equation to find the probability that all five tetrahedrons land on 3.

**25. Error Analysis** Two students find the probability of rolling two number cubes and getting a sum less than 6. Which student is correct? Explain the error.

<table>
<thead>
<tr>
<th>Student A</th>
<th>Student B</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{10}{36} = \frac{5}{18})</td>
<td>(\frac{15}{36} = \frac{5}{12})</td>
</tr>
</tbody>
</table>

**26. Multi-Step** A game has two spinners. After spinning both spinners, the sum of the spins is found.

- Make a table of all possible outcomes.
- What is the probability that the sum is greater than 8?

**27. Multiple Choice** What is the first step in solving the inequality \(2(x + 5) > x + 12\)?
A. Combine the variables.
B. Use the Addition Property of Inequality.
C. Apply the Distributive Property.
D. Use the Multiplication Property of Inequality.

**28. Travel** A car rental company charges $40 a day with no additional mileage fees. Another company charges $24 each day plus $0.16 per mile. How many miles would have to be driven in one day for the first company to offer the better deal?

**29. Write** Explain how to solve the inequality \(2x + 5 > -3(x - 15)\). Identify the solution.

**30. Estimate** During their freshman year, Malcolm averaged 17.3 points per game and Frederico averaged 15.2 points per game. In their sophomore year, Malcolm averaged 19.1 points per game and Frederico averaged 18.4 points per game. If the trend continues, in which years will Frederico have a better average than Malcolm?
Solving Multi-Step Compound Inequalities

Warm Up

1. **Vocabulary** A(n) _______ is made up of two inequalities combined with the word **and** or **or**.

Solve each inequality.

2. \(8x > 6x - 12\)

3. \(-1 - (-7) \leq 3(y - 6)\)

4. \(x - 2 < -7 \text{ OR } 2x \geq 11\)

5. **Multiple Choice** Which compound inequality is equivalent to \(6 \leq -2x < 22\)?
   - A \(-11 < x \leq -3\)
   - B \(-11 > x \geq -3\)
   - C \(8 \leq x < 24\)
   - D \(-24 < x \leq -8\)

New Concepts

Inequalities can be solved in two or more steps using inverse operations. A compound inequality is made of two inequalities joined by the word **AND** or **OR**.

\[-2 < x \text{ AND } x \leq 5 \quad x \leq -4 \text{ OR } x \geq 1\]

**Example 1** Solving Multi-Step Compound Inequalities

Solve and graph each inequality.

**a.** \(4x - 7 < 3 \text{ OR } 2x - 19 > -7\)

**SOLUTION** Isolate the variable in both inequalities.

\[
4x - 7 < 3 \text{ OR } 2x - 19 > -7
\]

\[
+7 \quad +7 \quad +19 \quad +19
\]

\[
4x < 10 \text{ OR } 2x > 12
\]

\[
\frac{4x}{4} < \frac{10}{4} \text{ OR } \frac{2x}{2} > \frac{12}{2}
\]

\[
x < 2\frac{1}{2} \text{ OR } x > 6
\]

**b.** \(-9 \leq 3x - 4 + 2x \leq 11\)

**SOLUTION** Isolate the variable between the inequality signs.

\[
-9 \leq 3x - 4 + 2x \leq 11
\]

\[
+4 \quad +4 \quad +4
\]

\[
-5 \leq 3x + 2x \leq 15
\]

\[
\frac{-5}{5} \leq \frac{5x}{5} \leq \frac{15}{5}
\]

\[-1 \leq x \leq 3\]
**Example 2**  Simplifying Before Solving Inequalities

Solve the inequality. Justify each step.

**a.** \(-15 \leq 3(2x - 1) \leq 39\)

**SOLUTION**

\[-15 \leq 3(2x - 1) \leq 39\]

\[-15 \leq 6x - 3 \leq 39\]

\[+3 \quad +3 \quad +3\]

\[-12 \leq 6x \leq 42\]

\[\frac{1}{6} \cdot -12 \leq \frac{1}{6} \cdot 6x \leq \frac{1}{6} \cdot 42\]

\[-2 \leq x \leq 7\]

**b.** \(-12 \geq -6b - 18 \text{ OR } -2(4 - b) \geq 10\)

**SOLUTION**

\[-12 \geq -6b - 18 \text{ OR } -2(4 - b) \geq 10\]

\[-12 \geq -6b - 18 \text{ OR } -8 + 2b \geq 10\]

\[+18 \quad +18 \quad +8 \quad +8\]

\[6 \geq -6b \text{ OR } 2b \geq 18\]

\[6 \cdot -6 \leq -6b \text{ OR } \frac{2b}{2} \geq \frac{18}{2}\]

\[-1 \leq b \text{ OR } b \geq 9\]

**Example 3**  Application: Zoology

Zoologists randomly choose 5 zebras out of a herd of 20. Four zebras weigh 540 pounds, 550 pounds, 520 pounds, and 530 pounds, respectively. What could the weight of the fifth zebra be if the average weight of all 5 zebras is to be between 500 and 600 pounds?

**SOLUTION**  Set up a compound inequality representing the situation and solve.

<table>
<thead>
<tr>
<th>Minimum Weight</th>
<th>Mean Weight</th>
<th>Maximum Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>greater than or equal to 500</td>
<td>(\frac{540 + 550 + 520 + 530 + x}{5})</td>
<td>less than or equal to 600</td>
</tr>
</tbody>
</table>

\[500 \leq \frac{540 + 550 + 520 + 530 + x}{5} \leq 600\]

\[500 \cdot 5 \leq \frac{2140 + x}{5} \cdot 5 \leq 600 \cdot 5\]

\[2500 \leq 2140 + x \leq 3000\]

\[-2140 \leq -2140 \leq -2140\]

\[360 \leq x \leq 860\]

The fifth zebra’s weight could be between 360 lb and 860 lb, inclusive.
Lesson Practice

Solve and graph each inequality.

a. \(2x + 9 < 8\) OR \(3x + 3 > 12\).

b. \(24 \leq 2x + 8 < 36\).

Solve the inequality. Justify each step.

c. \(6 \leq 2(x + 12) < 12\)

d. \(-16 > 2(x - 2)\) OR \(27 < 3(x + 2)\)

e. Of 4 babies born in a hospital in 1 night, 3 have weights of 5.2 pounds, 6.3 pounds, and 7.5 pounds, respectively. What could be the weight of the fourth baby if the average of all their weights fall within 6 and 8 pounds?

Practice Distributed and Integrated

Factor completely.

1. \(2x^2 + 9xy + 7y^2\)  
2. \(-4m^2 + 8mn + 5n^2\)

Find the product.

3. \((\sqrt{3} - 12)^2\)  
4. \((2x + \sqrt{3})(2x - \sqrt{3})\)

Find the excluded values.

5. \(\frac{m - 6}{2m - 10}\)  
6. \(\frac{y + 4}{-2y - 6}\)

Solve the inequality. Then graph and check the solution.

7. \(2z - 6 \leq z\)

8. Solve and graph \(2x + 9 > -x + 18\).

Determine if the following systems of equations are consistent and independent, consistent and dependent, or inconsistent.

9. \(y = 10x - 2\)  
10. \(y = 3x\)

\(y = 10x + 8\)  
\(2y = 6x\)

11. Multi-Step The perimeter of a square area rug is 48 feet. What is the length of each side? Express your answer as a radical number.

12. Quilting A quilter uses a series of rectangular patterns to design quilt blocks. The area of the quilt block can be represented by the trinomial \(x^2 + 7x + 12\) cm². If he plans a quilt block with \(x\) having a value of 20 centimeters, what is the dimension of the longer side of the block?

13. Multi-Step A real number is less than 12 or is greater than 15.

a. Write a compound inequality that represents the situation.

b. Graph the solution.
14. **Estimate** What is the lesser value of $q$ in the solution set of $|q - 24.9| = 5.1$?

15. **Cell Phones** You pay $10 a month plus $0.30 per minute for your cell phone. You budget $20 each month for your bill. To find the maximum minutes you can use your phone, solve the inequality $10 + 0.3m \leq 20$.

16. **Probability** Write a rational function that expresses the following probability. Find the probability $y$ of randomly choosing a red marble out of a bag full of $x$ number of marbles that contains only one red marble.

17. **Verify** Show that $-8u^3y + 56u^4y - 80u^3y = -8u^3(y(u - 5)(u - 2))$.

18. **Multiple Choice** Which expression is the complete factored form of $3x^6 + 6x^5 - 45x^4$?

A. $3x^4(x - 3)(x + 5)$

B. $x^4(3x - 9)(x + 5)$

C. $x^4(x - 3)(3x - 15)$

D. $(3x - 9)(3x^3 + 5x^4)$

19. Use the graph to find the theoretical probability of receiving each grade.

<table>
<thead>
<tr>
<th>Grade</th>
<th>Frequency of Grades</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>8</td>
</tr>
<tr>
<td>C</td>
<td>4</td>
</tr>
<tr>
<td>D</td>
<td>4</td>
</tr>
<tr>
<td>E</td>
<td>2</td>
</tr>
</tbody>
</table>

20. **Biology** A Punnett square shows the probability distribution for genes. A short pea plant contributes two short genes, labeled “t.” A tall pea plant contributes two tall genes, labeled “T.” The plant will be short if it inherits the combination “tt.” “TT” means the plant will be tall. The combination “Tt” also results in a tall plant. What is the probability that this plant will be short? Explain your answer.

21. **Error Analysis** Two students use an equation to find the probability of getting heads on four coins and rolling a 2 or 3 on two number cubes labeled 1–6. Which student is correct? Explain the error.

<table>
<thead>
<tr>
<th>Student A</th>
<th>Student B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(4 \text{ heads and two 2 or 3}) = \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{6}\right) \cdot \left(\frac{1}{6}\right) = \frac{1}{20,736}$</td>
<td>$P(4 \text{ heads and two 2 or 3}) = \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{3}\right)^2 = \frac{1}{144}$</td>
</tr>
</tbody>
</table>

22. **Multi-Step** Veejay is throwing a party. It costs $75 to rent a skating arena plus $3 per person to rent skates. It costs $100 to rent a bowling alley plus $2 per person to rent bowling shoes. How many people would Veejay have to invite to his party for the bowling alley to cost more than the skating arena?

a. Write an inequality to answer.

b. Solve the inequality.

c. Explain the correct domain of the solution set.
23. **Music** Amber normally listens to 1 new CD and 7 old CDs every day. She starts to listen to 2 more new CDs each day and 1 less old CD each day. How many days will it take her to listen to more new CDs than old CDs?

24. **Geometry** The length of a rectangle is greater than its width. The length is $4x + 7$ and the width is $5x - 2$. What does the value of $x$ have to be for this statement to be true?

25. **Error Analysis** Two students are told to write an inequality that is a contradiction. Which student is correct? Explain the error.

<table>
<thead>
<tr>
<th>Student A</th>
<th>Student B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2 + x &gt; x + 3$</td>
<td>$2x + 24 &lt; 3x + 24$</td>
</tr>
</tbody>
</table>


- $17 > -2x - 7$ OR $27 > 3(x + 6)$

27. **Multiple Choice** What is the solution to $32 < 7x + 11 < 39$?

- A $21 > x > 28$
- B $3 < x < 4$
- C $3 < x < 4$
- D $3 > x > 4$

28. **Cholesterol Levels** An average level of HDL, a type of good cholesterol, for a person is usually no more than 60, and an unhealthy level is lower than 40. A doctor sees 4 patients and tests their HDL levels. The first 3 levels are 45, 52, and 60. What can the fourth patient’s HDL level be if the average of all four patients’ levels fall in the average, but not unhealthy, range of levels?

29. **Formulate** Half of Mr. Rubenstein’s math class studied for the test and the other half did not. Everyone who studied for the test got a score no lower than 90; everyone who did not study got a score lower than 70. Write the scores of the class as an inequality.

30. **Estimate** Felipe wants to earn a grade between 90 and 100 in math. There are 4 major tests over the year, which are averaged to determine his final grade. Felipe scored 94, 88, and 91 on the first 3 tests. What must he score on the last test for his average grade to fall between 90 and 100? Round his scores to solve.
Factoring Special Products

Warm Up

1. Vocabulary A trinomial that is the square of a binomial is called a(n) ___________.

Factor.

2. \(3x^4 - 12x\)
3. \(48y^2 + 16y^3 - 56y^5\)

Multiply.

4. \((2b - 3)^2\)
5. \((3x + 7)(3x - 7)\)

New Concepts

Look for a pattern in the products.

\[
\begin{align*}
(x + 1)^2 &= (x + 1)(x + 1) = x^2 + 2x + 1 &\rightarrow& x^2 + 2 \cdot 1x + 1^2 \\
(x + 2)^2 &= (x + 2)(x + 2) = x^2 + 4x + 4 &\rightarrow& x^2 + 2 \cdot 2x + 2^2 \\
(x + 3)^2 &= (x + 3)(x + 3) = x^2 + 6x + 9 &\rightarrow& x^2 + 2 \cdot 3x + 3^2 \\
(x - 1)^2 &= (x - 1)(x - 1) = x^2 - 2x + 1 &\rightarrow& x^2 - 2 \cdot 1x + (-1)^2 \\
(x - 2)^2 &= (x - 2)(x - 2) = x^2 - 4x + 4 &\rightarrow& x^2 - 2 \cdot 2x + (-2)^2
\end{align*}
\]

The pattern is:

Square the first term in the binomial.
Square the second term in the binomial.
Multiply the product of both terms by 2.

Recall that a perfect-square trinomial is a polynomial that is the square of a binomial. The trinomial has the form \(a^2 + 2ab + b^2\) or \(a^2 - 2ab + b^2\). When squaring binomials use the following patterns:

\[
(a + b)^2 = a^2 + 2ab + b^2 \\
(a - b)^2 = a^2 - 2ab + b^2
\]

Use the same patterns to factor perfect-square trinomials.

Perfect-Square Trinomials

The factored form of a perfect-square trinomial is:

\[
\begin{align*}
a^2 + 2ab + b^2 &= (a + b)^2 \quad \text{Example: } x^2 + 12x + 36 = (x + 6)^2 \\
a^2 - 2ab + b^2 &= (a - b)^2 \quad \text{Example: } x^2 - 12x + 36 = (x - 6)^2
\end{align*}
\]

Online Connection

www.SaxonMathResources.com
Example 1  Factoring Perfect-Square Trinomials

Determine whether each polynomial is a perfect-square trinomial. If it is, factor the trinomial.

\( a. \)  \( x^2 + 6x + 9 \)

**SOLUTION**

\[ x^2 + 6x + 9 = x^2 + 2 \cdot 3x + 3^2 \]

Write in perfect-square trinomial form.

\[ = (x + 3)^2 \]

It is a perfect-square trinomial.

\( b. \)  \( x^2 - 2x + 4 \)

**SOLUTION**

\[ x^2 - 2x + 4 \neq x^2 - 2 \cdot 1x + 2^2 \]

This is not equivalent to the perfect-square trinomial form. It is not a perfect-square trinomial.

\( c. \)  \( 36x^2 - 48x + 16 \)

**SOLUTION**

\[ 36x^2 - 48x + 16 = 4(9x^2 - 12x + 4) \]

Factor out 4.

\[ = 4[(3x)^2 - 2 \cdot (3x)(2) + 2^2] \]

Write in perfect-square trinomial form.

\[ = 4(3x - 2)^2 \]

It is a perfect-square trinomial.

Example 2  Application: Cell Phone Towers

A cellular phone tower’s signal covers a circular area with a radius \( r \) in miles. The strength of the signal is increased, and now covers an area of \( \pi r^2 + 10\pi r + 25\pi \) square miles. By how much did the radius of the coverage area increase?

**SOLUTION**

Factor the expression for the new coverage area.

\[ \pi r^2 + 10\pi r + 25\pi \]

\[ = \pi(r^2 + 10r + 25) \]

Factor \( \pi \) out of the expression.

\[ = \pi(r^2 + 2 \cdot 5r + 5^2) = \pi(r + 5)^2 \]

Write in perfect-square trinomial form.

The radius of the new circle is \( r + 5 \).

The radius of the coverage area increased by 5 miles.
Look for a pattern in the products.

\[(x + 1)(x - 1) = x^2 - 1 \quad \rightarrow \quad (x \cdot x) - 1x + 1x - (1 \cdot 1) = x^2 - 1^2\]
\[(x + 2)(x - 2) = x^2 - 4 \quad \rightarrow \quad (x \cdot x) - 2x + 2x - (2 \cdot 2) = x^2 - 2^2\]
\[(x + 3)(x - 3) = x^2 - 9 \quad \rightarrow \quad (x \cdot x) - 3x + 3x - (3 \cdot 3) = x^2 - 3^2\]
\[(a + b)(a - b) = a^2 - b^2 \quad \rightarrow \quad (a \cdot a) - ab + ab - (b \cdot b) = a^2 - b^2\]

The pattern can be used to factor the difference of two squares.

<table>
<thead>
<tr>
<th>Difference of Two Squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>The factored form of a difference of two squares is:</td>
</tr>
<tr>
<td>[a^2 - b^2 = (a + b)(a - b)]  Example: [x^2 - 49 = (x + 7)(x - 7)]</td>
</tr>
</tbody>
</table>

**Example 3**  Factoring the Difference of Two Squares

Determine whether each binomial is the difference of two squares. If so, factor the binomial.

**a.** \[4x^2 - 25\]

**SOLUTION**

\[4x^2 - 25\]
= \[(2 \cdot 2)(x \cdot x) - (5 \cdot 5)\]  Factor each term.
= \[(2x)^2 - 5^2\]  Write as a difference of two squares.
= \[(2x + 5)(2x - 5)\]  Factor.

**b.** \[9m^4 - 16n^6\]

**SOLUTION**

\[9m^4 - 16n^6\]
= \[(3 \cdot 3)(m^2 \cdot m^2) - (4 \cdot 4)(n^3 \cdot n^3)\]  Factor each term.
= \[(3m^2)^2 - (4n^3)^2\]  Write as a difference of two squares.
= \[(3m^2 + 4n^3)(3m^2 - 4n^3)\]  Factor.

**c.** \[x^2 - 8\]

**SOLUTION**

\[x^2 - 8\]
= \[(x \cdot x) - (4 \cdot 2)\]  Factor each term.
= \[x^2 - 8\]  This is not a difference of two squares.

**d.** \[-64 + z^8\]

**SOLUTION**

\[-64 + z^8 = z^8 - 64\]  Write terms in descending order.
= \[(z^4 \cdot z^4) - (8 \cdot 8)\]  Factor each term.
= \[(z^4)^2 - 8^2\]  Write as a difference of two squares.
= \[(z^4 + 8)(z^4 - 8)\]  Factor.
Example 4  Application: Garden Planning

Ganesh is designing a square border around a square pond. The area of the pond is 225 square feet. Use the difference of two squares to write an expression that represents the area of the border.

\[ \text{SOLUTION} \]

Area of larger square: Area of pond:
\[ A = lw \quad A = lw \]
\[ = (2f)(2f) \quad = 225 = (15)(15) \]

Find the difference between the area of the square surrounding the pond and the area of the pond.
\[ (2f)^2 - (15)^2 \]
Write as a difference of two squares.
\[ = (2f - 15)(2f + 15) \]
Factor.

The area of the border is \((2f - 15)(2f + 15)\) ft\(^2\).

Lesson Practice

Determine whether the polynomial is a perfect-square trinomial. If so, factor the trinomial.

\begin{itemize}
  \item \textbf{a.} \(x^2 + 14x + 49\)
  \item \textbf{b.} \(6n^4 - 12n^2 + 6\)
  \item \textbf{c.} \(3g^2 + 9g + 9\)
  \item \textbf{d.} A radio tower’s signal covers a circular area with a radius \(r\) in miles. The strength of the signal is increased, and now covers an area of \(\pi r^2 + 12\pi r + 36\pi\) square miles. By how much did the radius of the coverage area increase?
\end{itemize}

Determine whether the binomial is the difference of two squares. If so, factor the binomial.

\begin{itemize}
  \item \textbf{e.} \(25x^2 - 4\)
  \item \textbf{f.} \(9h^2 - 100a^2\)
  \item \textbf{g.} \(x^2 - 14\)
  \item \textbf{h.} \(-81 + x^{10}\)
  \item \textbf{i.} A square border is designed around a square pool. Use the difference of two squares to write and factor an expression that represents the area of the border.
\end{itemize}
Find the product.

1. \((7 + \sqrt{6})(4 - \sqrt{9})\)  
2. \((x + \sqrt{12})(x - \sqrt{3})\)

Solve.

3. \(-b + \frac{3}{8} \geq \frac{3b - 5}{8}\)
4. \(11h + 9 \leq 5h - 21\)

Factor completely.

5. \(3x^3 - 3x^4 - 216x^3\)
6. \(-12x^3 - 48x\)

Determine whether the polynomial is a perfect-square trinomial or a difference of two squares. Then factor the polynomial.

7. \(x^2 + 10x + 25\)
8. \(x^2 + 12x + 36\)

9. **Geometry** Show that \(TUV\) is a right triangle.

10. **Analyze** The expression \(-7x + 2y\) is one factor of a difference of two squares. What is the expanded polynomial?

11. **Hiking** Raul and his friends are hiking a 4-mile trail. After 2 hours of hiking, they turn off of the path to find a spot for lunch, and then hike back to the trail and continue to the end. Write an inequality to represent \(x\), the total distance they hiked.

Determine if the following systems are consistent and independent, consistent and dependent, or inconsistent.

12. \(y = 2x + 5\) \(y - 2x = 1\)
13. \(3y = 2x + 4\) \(3x = 4.5y - 6\)

14. Find the probability of rolling a sum of 10 or a set of doubles with two number cubes.
15. **Multi-Step** The local bank offers a savings account with a 3% annual interest rate. Alfred wants to earn at least $60 in interest. How much should he deposit?
   a. Write an inequality to represent the situation.
   b. Solve the inequality.
   c. How much should he deposit?
   d. Graph the solution.

16. **Food Packaging** The label of a certain cheese states that it weighs 8 ounces. The actual weight of the product sold is allowed to be 0.2 ounces above or below that. Write a compound inequality that represents this situation.

17. **Multi-Step** Solve \( \frac{|x + 3|}{4} = 6 \).
   a. Isolate the absolute-value expression.
   b. Use the definition of absolute value to rewrite the absolute-value equation as two equations.
   c. What is the solution set?

18. **Generalize** Explain how you know if the second terms of binomial factors are both positive, both negative, or have opposite signs.

19. **Consumerism** A preschool has a budget of $1000 to buy new outside toys. They will receive 1 free toy when they place the order. The number of toys, \( y \), that they can get is given by \( y = \frac{1000}{x} + 1 \), where \( x \) is the price per toy.
   a. What is the horizontal asymptote of this rational function?
   b. What is the vertical asymptote?
   c. If the price per toy is $5, how many toys will they receive?

20. **Measurement** The length of the hypotenuse of a right triangle is found by adding the squares of the two legs and then taking the square root. The sum of the squares of the legs is \( 16m^6 + 320m^5 + 1600m^4 \). Find the length of the hypotenuse by factoring.

21. **Verify** The table shows that the theoretical probability of landing on heads two times when flipping two coins is \( \frac{1}{4} \). Use an equation to show that the probability is correct.

<table>
<thead>
<tr>
<th></th>
<th>Tails</th>
<th>Heads</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tails</td>
<td>TT</td>
<td>TH</td>
</tr>
<tr>
<td>Heads</td>
<td>HT</td>
<td>HH</td>
</tr>
</tbody>
</table>

22. **Multiple Choice** Using the table, what is the probability of rolling an even number and spinning a yellow or a green?

<table>
<thead>
<tr>
<th></th>
<th>Red</th>
<th>Yellow</th>
<th>Green</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1R</td>
<td>1Y</td>
<td>1G</td>
</tr>
<tr>
<td>2</td>
<td>2R</td>
<td>2Y</td>
<td>2G</td>
</tr>
<tr>
<td>3</td>
<td>3R</td>
<td>3Y</td>
<td>3G</td>
</tr>
<tr>
<td>4</td>
<td>4R</td>
<td>4Y</td>
<td>4G</td>
</tr>
<tr>
<td>5</td>
<td>5R</td>
<td>5Y</td>
<td>5G</td>
</tr>
<tr>
<td>6</td>
<td>6R</td>
<td>6Y</td>
<td>6G</td>
</tr>
</tbody>
</table>

23. **Write** Describe how factoring can help find \( 45^2 - 15^2 \).
**24. Error Analysis** Beth claims that the inequality \(12x + 67 \geq 52 + 5x + 15\) is always true for any value of \(x\). Is Beth correct? Explain the error.

**25. Finances** Chad works 20 hours each week. Juan works 10 hours and makes an additional $50 in tips each week. They both get paid the same amount per hour. How much money do they each have to earn per hour for Chad to make more money than Juan?

**26. Error Analysis** Two students solve the following compound inequality. Which student is correct? Explain the error.

<table>
<thead>
<tr>
<th>Student A</th>
<th>Student B</th>
</tr>
</thead>
<tbody>
<tr>
<td>[10 &lt; -2x + 2 &lt; 16]</td>
<td>[10 &lt; -2x + 2 &lt; 16]</td>
</tr>
<tr>
<td>[-4 &lt; x &lt; -7]</td>
<td>[-4 &gt; x &gt; -7]</td>
</tr>
</tbody>
</table>

**27. Multi-Step** Yvonne learns that a refrigerator should be kept at a temperature of no more than 40° F but warmer than 32° F.

a. Write an expression to show the possible range of proper refrigerator temperatures.

b. Yvonne tests the temperatures of some refrigerators at an appliance store. The first 4 temperatures are 35°, 40°, 20°, and 45°. What should the last temperature be if the average of the temperatures is within the proper temperature range?

**28. Justify** Solve the inequality \(28 < 2(x + 3) < 42\) and justify each step.

**29. Multiple Choice** Which expression is a perfect-square trinomial?

- A \(9x^2 + 49\)
- B \(64x^2 - 100\)
- C \(6x^2 + 48x - 96\)
- D \(49x^2 - 28x + 4\)

**30. Home Improvement** A square storage shed sits in the corner of a square deck that has a side length of \(s\) feet. The shed has a side length of 8 feet. Harper wants to apply a coat of paint to the deck. Write and factor an expression to find the area of the deck Harper will paint, not including the storage shed.
1. **Vocabulary** The equation \( f(x) = 7x^2 - 3x + 1 \) is written in _________ (expanded, function) notation.

**Evaluate.**

2. \( 6x^3 \) for \( x = 2 \)

3. \( x^2 - 4x + 3 \) for \( x = -3 \)

4. \( 500 - 7x^2 \) for \( x = -10 \)

5. **Multiple Choice** Solve \( 7x - y = 2 + 6x \) for \( y \).

   A  \( y = \frac{7x - 2}{6} \)  
   B  \( y = x - 2 \)  
   C  \( y = -x \)  
   D  \( y = 13x + 2 \)

**New Concepts**

A function pairs each value in the domain with exactly one value in the range. A **quadratic function** is a function that can be written in the form \( f(x) = ax^2 + bx + c \), where \( a \) is not equal to 0. So, quadratic functions must have a quadratic term, but they may also have a linear and/or a constant term.

\[
f(x) = ax^2 + bx + c, \text{ where } a \neq 0
\]

- **quadratic term**
- **linear term**
- **constant term**

All quadratic functions consist of a polynomial expression with a degree of exactly 2. The degree of a polynomial is the same as the term with the greatest degree. The polynomial can be named by its highest degree.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Degree</th>
<th>Name Using Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3x + 2 )</td>
<td>1</td>
<td>Linear</td>
</tr>
<tr>
<td>( x^2 + 4x - 5 )</td>
<td>2</td>
<td>Quadratic</td>
</tr>
<tr>
<td>( 2x^3 - x^2 + 1 )</td>
<td>3</td>
<td>Cubic</td>
</tr>
</tbody>
</table>

A quadratic function can be written in many ways; however, there is a standard way to write a quadratic function.

<table>
<thead>
<tr>
<th>Standard Form of a Quadratic Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>The standard form of a quadratic function is ( f(x) = ax^2 + bx + c ), where ( a, b, ) and ( c ) are real numbers and ( a \neq 0 ).</td>
</tr>
</tbody>
</table>

If a function cannot be written in the standard form of a quadratic function, then the function is not quadratic.
Example 1  Identifying Quadratic Functions

Determine whether each function represents a quadratic function.

a. \( y + 7x = 4x^2 - 6 \)

SOLUTION

\[
y + 7x = 4x^2 - 6 \\
y = 4x^2 - 7x - 6
\]

Solve for \( y \).

It is a quadratic function because it can be written in the standard form of a quadratic equation.

b. \( y = 5 + 2x \)

SOLUTION

\[
y = 5 + 2x
\]

Since there is no quadratic term, it is not a quadratic function.

c. \( -2x^3 + y = -5x^3 + x^2 \)

SOLUTION

\[
-2x^3 + y = -5x^3 + x^2 \\
y = -3x^3 + x^2
\]

Add \( 2x^3 \) to both sides.

Since there is a cubic term, it is not a quadratic function.

The graph of \( f(x) = x^2 \) is known as the quadratic parent function. Graph the parent function by making a table of values. Plot the points and connect them with a smooth U-shaped curve called a parabola.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-2</th>
<th>0</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>16</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>16</td>
</tr>
</tbody>
</table>

Example 2  Graphing Quadratic Functions Using a Table

Use a table to graph the function.

\( f(x) = -3x^2 \)

SOLUTION

Plot the points in a coordinate plane and draw a smooth curve through the points.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-12</td>
<td>-3</td>
<td>0</td>
<td>-3</td>
<td>-12</td>
</tr>
</tbody>
</table>
The direction of a parabola can be determined by value of the coefficient of the quadratic term.

<table>
<thead>
<tr>
<th>Direction of a Parabola</th>
</tr>
</thead>
<tbody>
<tr>
<td>For a quadratic function in standard form, (y = ax^2 + bx + c):</td>
</tr>
<tr>
<td>If (a &lt; 0), the parabola opens downward.</td>
</tr>
<tr>
<td>If (a &gt; 0), the parabola opens upward.</td>
</tr>
</tbody>
</table>

**Example 3  Determining the Direction of a Parabola**

Determine whether the graph of each function opens upward or downward.

a. \(f(x) = 3x^2 + 8\)

**SOLUTION**

\(f(x) = 3x^2 + 8\) \hspace{1cm} \(a = 3\)

The graph opens upward because \(a > 0\).

b. \(f(x) = 3x - x^2 + 5\)

**SOLUTION**

\(f(x) = 3x - x^2 + 5\)

\(f(x) = -x^2 + 3x + 5\) \hspace{1cm} Write in standard form.

Since \(a = -1\), \(a < 0\) and the graph opens downward.

**Example 4  Application: Free Fall**

A pebble is dropped from a 256-foot-tall cliff. The equation \(256 - h = 16t^2\) can be used to find the height \(h\) of the pebble after falling for \(t\) seconds.

Find the height of the pebble after falling for 2 seconds.

**SOLUTION**

**Understand** Determine the height of the pebble using the function \(256 - h = 16t^2\). Define the variables in the function.

\(h = \text{height in feet} \hspace{1cm} t = \text{time in seconds}\)

**Plan** Solve the equation for height \(h\), and then find \(h\) when \(t = 2\).

**Solve** Solve the equation for height \(h\).

\(256 - h = 16t^2\)

\(h = -16t^2 + 256\)

Find \(h\) when \(t = 2\).

\(h = -16(2)^2 + 256\)

\(= -16(4) + 256\)

\(= -64 + 256\)

\(= 192\) feet

The height of the pebble after falling for 2 seconds is 192 feet.
Check Make a table of values. Choose positive values for the number of seconds $t$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>256</td>
<td>240</td>
<td>192</td>
<td>112</td>
<td>0</td>
</tr>
</tbody>
</table>

From the table the range is $0 \leq h \leq 256$ and the answer was 192 feet, so the answer is reasonable.

Lesson Practice

Determine whether each function represents a quadratic function.

1. $4 - y = x - 2x^2 - 3$
2. $x = -x^2 + y$
3. $4 = y$

Use a table of values to graph the function.

4. $f(x) = 4x^2 - 3$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$2$</td>
<td></td>
</tr>
</tbody>
</table>

Determine whether the graph of each function opens upward or downward.

5. $f(x) = 2x^2 - 4$
6. $f(x) = 2x - 5x^2$
7. An acorn falls from a 16-foot-tall oak tree. The equation $h = -16t^2 + 16$ can be used to find the height $h$ of the acorn after falling for $t$ seconds. Find the height of the acorn after falling for 0.5 seconds.

Practice Distributed and Integrated

Determine whether the polynomial is a perfect-square trinomial or a difference of two squares. Then factor the polynomial.

1. $q^2 + 18q + 81$
2. $36x^2 - 144$

Simplify.

3. $\sqrt{12} + \sqrt{48} - \sqrt{27}$
4. $\sqrt{18} + \sqrt{32} + \sqrt{50}$

Solve. Graph the solution.

5. $2p + 7 > p - 10$
6. $16 < 2x + 8$ OR $15 > 7x + 1$

*7. Rewrite $x + 15x^2 - y = 4$ in the standard form of a quadratic function, if possible.
Find the probability of the following events.

8. spinning a blue section or a letter B

9. spinning a gray section or a letter D

10. spinning a white section or a letter C

11. **Computer Electronics** The failure rate for desktop computers is 5% for the first year of use. For notebooks it is 15%. If you purchase both a new computer and a notebook, what is the probability that either one will fail in the first year?

12. **Multi-Step** Use the scatter plot.
   a. Using two points on the line, find an equation for the trend line.
   b. Does the graph show a positive correlation, a negative correlation, or no correlation?

13. **Entertainment** People are often employed by amusement parks to predict the ages and weights of patrons. For a fee, one guesser claims she can predict a patron’s weight within three pounds of the correct weight. If the guess is incorrect, the patron receives a prize. Write and solve an absolute-value equation for the maximum and minimum values of a correct guess for a person weighing 162 pounds.

14. **Multi-Step** The success rate on an exam is represented by \(72x^2 - 156x + 72\).
   a. Evaluate the expression for \(x = 2\).
   b. Factor the expression completely.

15. **Verify** Show that \(\sqrt{14} \cdot \sqrt{21} = 7\sqrt{6}\).

16. Find the vertical asymptote: \(y = \frac{1.6}{x + 2.5 + 7.8}\).

17. **Baseball** A pop fly is hit into the infield. In the expression \(-5t^2 + 40t - 35\), \(t\) represents the time that the ball was 37 meters high. Factor the expression completely.

18. One student is selected from a school committee that has 12 seniors, 8 juniors, 10 sophomores, and 4 freshmen. Make a graph showing the frequency distribution.

19. **Data Analysis** Use the frequency distribution from the table to make a bar graph.

<table>
<thead>
<tr>
<th>Pasta Salad</th>
<th>Cucumber Salad</th>
<th>Caesar Salad</th>
<th>Carrot Salad</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>12</td>
<td>16</td>
<td>22</td>
</tr>
</tbody>
</table>

*20. The value of \(a\) varies jointly with \(b\) and \(c\). What is the constant of variation if \(a = 18\), \(b = 2\), and \(c = 3\)? Write an equation expressing the given relationship.

21. **Model** Graph the solution for \(2(x + 9) - 14 > 3x + 7 + 2x\).
22. **Multiple Choice** What is the justification for subtracting 11 from all parts of the inequality $32 < 7x + 11 < 39$?
   A Combine the variable.
   B Addition Property of Inequality
   C Distributive Property of Inequality
   D Multiplication Property of Inequality

23. **Health Checks** A borderline unhealthy cholesterol level is between 200 and 240. Five patients come to the doctor with borderline cholesterol levels. The first 4 have levels of 210, 230, 225, and 235. What could the fifth patient’s level be if the average of all the patients’ levels are within the borderline unhealthy range?

24. **Error Analysis** Ms. Cho asks two students to factor the polynomial only if it is a perfect-square trinomial. Which student is correct? Explain the error.

   **Student A**
   
   $x^2 + 8x - 16 = (x - 4)^2$

   **Student B**
   
   $x^2 + 8x - 16$ is not a perfect-square trinomial.

25. **Multi-Step** A cylindrical thermos has a radius of $r$. Beneath the outer surface is an insulating layer. The volume, in cubic centimeters, that the thermos can hold is given by the expression $30\pi r^2 - 60\pi r + 30\pi$.
   a. Factor the polynomial representing the volume of the thermos.
   b. How thick is the insulating layer?
   c. What is the height of the thermos?

26. **Geometry** The surface area of a cube is given by the expression $6x^2 + 36x + 54$. What is the length of one side in terms of $x$?

27. **Multiple Choice** Which function does the table of values represent?
   A $y = x + 8$
   B $y = -x^2 + 12$
   C $y = x^2 + 2$
   D $y = -x^2 - 3x + 4$

28. **Formulate** Write the equation of a function with degree 1 and of a function with degree 2.

29. **Generalize** What is the relationship of the graph of $y = x^2$ to the graph of $y = -x^2$?

30. **Water Fountains** A circular fountain sits in front of a city library. A pool of water surrounds a sculpture that sits on a circular platform in the middle. The radius of the sculpture is half the radius of the entire fountain. Write an equation to represent the area of the pool. Is the equation a quadratic function?
Warm Up

1. **Vocabulary** The square of an integer is a ________ (perfect square, radical expression).

Simplify.

2. \(\sqrt{625}\)  
3. \(\sqrt{196}\)  
4. \(\sqrt{216}\)

Estimate to the nearest tenth.

5. \(\sqrt{389}\)

New Concepts

The Pythagorean Theorem states an important relationship among the lengths of the sides of any right triangle.

<table>
<thead>
<tr>
<th>Math Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>The <strong>hypotenuse</strong> of a right triangle is the side opposite the right angle. The <strong>legs</strong> are the sides that form the right angle.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Pythagorean Theorem</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>If a triangle is a right triangle with legs of lengths (a) and (b) and hypotenuse of length (c), then (a^2 + b^2 = c^2).</td>
</tr>
</tbody>
</table>

**Exploration** Justifying the Pythagorean Theorem

The legs of the four blue congruent triangles form a square. The gray quadrilateral is also a square.

1. Explain why \((a + b)^2\) represents the area of the outer square formed by the blue triangles.

2. What area does the expression \(\frac{1}{2}ab\) represent?

3. Write an algebraic expression for the area of the gray square.

4. Use the expressions from problems 1, 2, and 3 to translate the statement below into an equation.

   Area of outer square = Area of 4 triangles + Area of gray square

5. Show that the equation you wrote in problem 4 simplifies to \(a^2 + b^2 = c^2\).
Example 1  Calculating Missing Side Lengths

Use the Pythagorean Theorem to find the missing side lengths.

a. Find side length \( c \).

**SOLUTION**

\[ a^2 + b^2 = c^2 \]  \hspace{1cm} \text{Pythagorean Theorem}

\[ 8^2 + 6^2 = c^2 \]  \hspace{1cm} \text{Substitute 8 for } a \text{ and 6 for } b.

\[ 64 + 36 = c^2 \]  \hspace{1cm} \text{Simplify.}

\[ 100 = c^2 \]  \hspace{1cm} \text{Add.}

\[ \sqrt{100} = c \]  \hspace{1cm} \text{Take the square root of each side.}

\[ 10 = c \]  \hspace{1cm} \text{Simplify. Because } c \text{ is a length, } c \text{ cannot be negative.}

The side length \( c \) is 10.

b. Find side length \( t \) to the nearest tenth.

**SOLUTION**

\[ a^2 + b^2 = c^2 \]  \hspace{1cm} \text{Pythagorean Theorem}

\[ 4^2 + t^2 = 7^2 \]  \hspace{1cm} \text{Substitute 4 for } a, \text{ t for } b, \text{ and 7 for } c.

\[ 16 + t^2 = 49 \]  \hspace{1cm} \text{Simplify.}

\[ t^2 = 33 \]  \hspace{1cm} \text{Subtract 16 from each side.}

\[ t = \sqrt{33} \]  \hspace{1cm} \text{Take the positive square root of each side.}

\[ t \approx 5.7 \]  \hspace{1cm} \text{Estimate; round to the nearest tenth.}

c. Find side length \( k \).

**SOLUTION**

\[ a^2 + b^2 = c^2 \]  \hspace{1cm} \text{Pythagorean Theorem}

\[ k^2 + 5^2 = (\sqrt{61})^2 \]  \hspace{1cm} \text{Substitute } k \text{ for } a, 5 \text{ for } b, \text{ and } \sqrt{61} \text{ for } c.

\[ k^2 + 25 = 61 \]  \hspace{1cm} \text{Simplify.}

\[ k^2 = 36 \]  \hspace{1cm} \text{Subtract 25 from each side.}

\[ k = \sqrt{36} \]  \hspace{1cm} \text{Take the positive square root of each side.}

\[ k = 6 \]  \hspace{1cm} \text{Simplify.}

d. Find side length \( m \) in simplest radical form.

**SOLUTION**

\[ a^2 + b^2 = c^2 \]  \hspace{1cm} \text{Pythagorean Theorem}

\[ 12^2 + 8^2 = m^2 \]  \hspace{1cm} \text{Substitute 12 for } a, 8 \text{ for } b, \text{ and } m \text{ for } c.

\[ 208 = m^2 \]  \hspace{1cm} \text{Simplify the left side.}

\[ \sqrt{208} = m \]  \hspace{1cm} \text{Take the positive square root of each side.}

\[ 4\sqrt{13} = m \]  \hspace{1cm} \text{Simplify the square root.}
The Converse of the Pythagorean Theorem is also true; that is, if a triangle has side lengths \(a\), \(b\), and \(c\) that satisfy the equation \(a^2 + b^2 = c^2\), then the triangle is a right triangle with legs of lengths \(a\) and \(b\) and hypotenuse of length \(c\).

<table>
<thead>
<tr>
<th>Pythagorean Triples</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Pythagorean triple is a group of three nonzero whole numbers (a), (b), and (c) that represent the lengths of the sides of a right triangle. Two triangles whose side lengths are Pythagorean triples are shown below.</td>
</tr>
</tbody>
</table>

![Pythagorean Triples Diagram](image)

### Example 2 Determining a Right Triangle

Determine whether the given side lengths form a Pythagorean triple.

**a.** 9, 40, 41

**SOLUTION**

Check whether 9, 40, and 41 satisfy the converse of the Pythagorean Theorem.

\[
9^2 + 40^2 \neq 41^2 \quad \text{Substitute } 9, 40, \text{ and } 41 \text{ into } a^2 + b^2 = c^2.
\]

\[
81 + 1600 \neq 1681 \quad \text{Simplify.}
\]

\[
1681 = 1681 \, \checkmark \quad \text{The equation is true.}
\]

Because 9, 40, and 41 are three nonzero whole numbers that satisfy \(a^2 + b^2 = c^2\), they form a Pythagorean triple.

**b.** 8, 10, 12

**SOLUTION**

Check whether 8, 10, and 12 satisfy the converse of the Pythagorean Theorem.

\[
8^2 + 10^2 \neq 12^2 \quad \text{Substitute } 8, 10, \text{ and } 12 \text{ into } a^2 + b^2 = c^2.
\]

\[
64 + 100 \neq 144 \quad \text{Simplify.}
\]

\[
164 \neq 144 \quad \text{The equation is false.}
\]

Because 8, 10, and 12 do not satisfy \(a^2 + b^2 = c^2\), they do not form a Pythagorean triple.

**c.** 7, 11, \(\sqrt{170}\)

**SOLUTION**

Because \(\sqrt{170}\) is not a nonzero whole number, the lengths 7, 11, and \(\sqrt{170}\) do not form a Pythagorean triple.
Example 3  Application: Length of a Ladder

The ladder in the diagram satisfies the “1 in 4 rule,” a rule of thumb for the safe use of ladders. This rule states that when the bottom of a ladder is positioned $x$ feet from the base of a building, the top of the ladder should reach a point $4x$ feet off the ground. Find the length of the ladder in the diagram. Round your answer to the nearest tenth of a foot.

SOLUTION

The ladder is the hypotenuse of a right triangle. The lengths of the legs of the triangle are 5 feet and 20 feet. Use the Pythagorean Theorem to find the length $c$ of the ladder.

\[ a^2 + b^2 = c^2 \]

\[ 5^2 + 20^2 = c^2 \]

Substitute 5 for $a$ and 20 for $b$.

\[ 425 = c^2 \]

Simplify the left side.

\[ \sqrt{425} = c \]

Take the positive square root of each side.

\[ 5\sqrt{17} = c \]

Simplify the square root.

\[ 20.6 \approx c \]

Estimate; round to the nearest tenth.

The length of the ladder is 20.6 feet.

Lesson Practice

Use the Pythagorean Theorem to find the missing side lengths. (Ex. 1)

a. Find side length $c$.

b. Find side length $m$ to the nearest tenth.

c. Find side length $r$.

d. Find side length $s$ in simplest radical form.
Determine whether the given side lengths form a Pythagorean triple.

- e. 5, 9, 11
- f. 8, 15, 17
- g. 4, $\sqrt{65}$, 13

**h. (Length of a Ladder)** A ladder leaning against a building satisfies the “1 in 4 rule” for the safe use of ladders. The bottom of the ladder is 8 ft from the base of the building. The top of the ladder touches the building 32 feet above the ground. Find the length of the ladder to the nearest tenth.

---

**Practice**  Distributed and Integrated

**Simplify.**

1. $3\sqrt{45} - \sqrt{5}$
2. $\frac{p^{-1}}{w} \left( \frac{wx}{cp^{-2}q^{-4}} + 5pq^{-3} \right)$

**Factor completely.**

3. $-3t^3 - 27t^2 - 24t$
4. $4x^4 - 16x^2$
5. $2x^2 + 14 - 9x - x^2$

**Determine whether the polynomial is a perfect-square trinomial or a difference of two squares. Then factor the polynomial.**

6. $3g^2 - 12$
7. $9x^2 - 24x + 16$

**Write the equations in the standard form of a quadratic function, if possible.**

8. $4 + y = -8 + 16x$
9. $y + x^2 = 3x^2 - 10x + 12$

**Solve.**

10. $0.7 + 0.05y = 0.715$
11. $\frac{1}{2} + \frac{3}{4}x = \frac{1}{6}x + 2$

12. Find the solution of $-1.2x \geq -4.8$. Then graph the solution set.

13. Write an equation for a line that passes through (1, 5) and is parallel to $y = -3\frac{1}{2}x - 9$.

14. (Gardening) Niko wants to build a fence around his garden. If his garden measures $2\sqrt{4}$ feet by $\sqrt{25}$ feet, how many feet of fencing does Niko need?

15. **Verify** Show that the solution to $\frac{x}{-5} + 6 \leq 10$ is $x \geq -20$. 

---

Saxon Algebra 1
16. Use the Pythagorean Theorem to find the missing side length. Give the answer in simplest radical form.

$$c$$

17. Find the vertical asymptote: \( y = \frac{2.4}{x + 4.5} + 6.9 \).

18. Given that \( y \) varies inversely with \( x \), identify the constant of variation when \( x = 4 \) and \( y = 2 \).

19. Statistics In 2005 the population of North Dakota was about 635,000—a decrease of about 7000 from five years earlier. The population of Wyoming in 2005 was 509,000—an increase of about 15,000 from five years earlier. If this trend continues, around what year will Wyoming’s population exceed North Dakota’s?

20. Sail Dimensions A main sail can be modeled by a right triangle whose sides are called the leach edge, the luff edge, and the foot. If the luff edge measures 27.5 feet and the foot measures 10 feet, use the Pythagorean Theorem to estimate the length of the leach edge to the nearest foot.

21. Multi-Step In cooking school, Larissa learns that some foods should never be kept in the “danger zone”: the temperature at which bacteria grow the fastest, potentially causing food poisoning. She learns that the danger zone is 5°C to 60°C.

   a. Write an inequality that shows the temperatures that are in the danger zone in degrees Celsius.

   b. Find an inequality that shows the temperatures that are in the danger zone in degrees Fahrenheit by substituting the expression \( \frac{5}{9}(f - 32) \) for the variable used in the inequality from part a, and then solve for \( f \).

   c. Write an inequality to show at what temperature food should be kept in degrees Fahrenheit.

22. Verify Solve the inequality \( 24 < 2x + 6 < 36 \). Check to make sure that the equation really is an AND inequality.
**23. Error Analysis** Two students factor the polynomial. Which student is correct? Explain the error.

<table>
<thead>
<tr>
<th>Student A</th>
<th>Student B</th>
</tr>
</thead>
<tbody>
<tr>
<td>(25x^2 - 36 = (5x - 6)^2)</td>
<td>(25x^2 - 36 = (5x - 6)(5x + 6))</td>
</tr>
</tbody>
</table>

**24. Tires** A truck’s tire has an outside radius of \(r\) inches. The area of the side of the tire, not including the inside rim, is \(\pi r^2 - 81\pi\) inches. What is the diameter of the rim?

**25. Geometry** Graph the quadratic function representing the total surface area of a cube with side length \(x\).

**26. Write** Explain why the Pythagorean Theorem cannot be used to find the missing side length \(c\).

\[\begin{array}{c}
12 \\
9 \\
\end{array}\]

**27. Formulate** One leg of a right triangle is twice the length of the other leg. The length of the hypotenuse is \(\sqrt{45}\) centimeters. Let \(x\) represent the length of the shorter leg. Use the Pythagorean Theorem to write and solve an equation to find the length of the legs.

**28. Multiple Choice** What is the perimeter of the triangle to the nearest inch?

- A 212 inches
- B 30 inches
- C 32 inches
- D 12 inches

**29. Grades** The grade a student earns on a project is represented by \(6x^2 - 11x + 35\), where \(x\) is the number of hours spent working on the project.

a. Factor the polynomial.

b. What grade is earned when a student works 4 hours on the project?

**30. Multi-Step** Write an expression for the area of the square. Then find the area.
Calculating the Midpoint and Length of a Segment

Warm Up

1. **Vocabulary** In an ordered pair, the ________ (x-coordinate, y-coordinate) indicates the distance up or down from the origin.

Simplify.

2. \(-3.8 - 5.5\)

3. \((-6 - (-3))^2\)

4. When viewed from the side, a skateboard landing ramp looks like a right triangle. What is the actual length of a ramp that is 2 feet tall if the base is 4 feet long? Round your answer to the nearest tenth.

5. **Multiple Choice** Identify the coordinates of point \(A\).

   - A \((4, 2)\)
   - B \((-4, 2)\)
   - C \((2, -4)\)
   - D \((4, -2)\)

New Concepts

The Pythagorean Theorem is used to find distances that are difficult to measure directly.

**Example 1** Calculating Distance Using the Pythagorean Theorem

The diagram shows a grid of city streets. A car travels from point \(P\) to point \(Q\) by moving east to point \(R\) and then south to point \(Q\). What is the direct distance (in city blocks) from point \(P\) to point \(Q\)?

**SOLUTION** To find the direct distance (in city blocks) from \(P\) to \(Q\), use the Pythagorean Theorem, written in the form \((PQ)^2 = (PR)^2 + (RQ)^2\).

\[
(PQ)^2 = (PR)^2 + (RQ)^2 \quad \text{Pythagorean Theorem}
\]
\[
(PQ)^2 = 4^2 + 3^2 \quad \text{Substitute 4 for } PR \text{ and 3 for } RQ.
\]
\[
\sqrt{(PQ)^2} = \sqrt{4^2 + 3^2} \quad \text{Take the positive square root of each side.}
\]
\[
(PQ)^2 = 25 \quad \text{Simplify under the radical.}
\]
\[
PQ = 5 \quad \text{Simplify the square root.}
\]

The direct distance from point \(P\) to point \(Q\) is 5 city blocks.
The Pythagorean Theorem can also be used to find the distance between two points in a coordinate plane. In the diagram for Example 1, let 1st Ave. be the \(x\)-axis and let A St. be the \(y\)-axis; then \(P\) is \((1, 4)\) and \(Q\) is \((5, 1)\). The lengths \(PR\) and \(RQ\) are found by subtracting coordinates.

\[
(PQ)^2 = (PR)^2 + (RQ)^2
\]
\[
\sqrt{(PQ)^2} = \sqrt{(PR)^2 + (RQ)^2}
\]
\[
\sqrt{(PQ)^2} = \sqrt{|5 - 1|^2 + |1 - 4|^2}
\]
\[
\sqrt{(PQ)^2} = \sqrt{4^2 + 3^2}
\]
\[
\sqrt{(PQ)^2} = \sqrt{16 + 9}
\]
\[
\sqrt{(PQ)^2} = \sqrt{25}
\]
\[
PQ = 5
\]

This method of finding the distance between two points leads to the distance formula.

### The Distance Formula

The distance \(d\) between two points \((x_1, y_1)\) and \((x_2, y_2)\) is

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\]

### Example 2 Finding the Distance Between Two Points

Find the distance between \((3, -2)\) and \((6, 4)\).

**SOLUTION** Use the distance formula. Substitute \((3, -2)\) for \((x_1, y_1)\) and \((6, 4)\) for \((x_2, y_2)\).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]
\[
d = \sqrt{(6 - 3)^2 + (4 - (-2))^2}
\]
\[
d = \sqrt{3^2 + 6^2}
\]
\[
d = \sqrt{9 + 36}
\]
\[
d = \sqrt{45}
\]
\[
d = 3\sqrt{5}
\]

The distance between \((3, -2)\) and \((6, 4)\) is \(3\sqrt{5}\).

### Example 3 Classifying Polygons

Determine whether quadrilateral \(ABCD\) is a rhombus.

A rhombus is a quadrilateral with four congruent sides.
Math Reasoning

Connect How is finding a midpoint related to finding an average?

Generalize Does it make a difference as to which point is \((x_1, y_1)\) or \((x_2, y_2)\)? Explain.

\[ \textbf{SOLUTION} \] Use the distance formula to find the length of each side of \(ABCD\).

\[
AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
= \sqrt{(2 - (-1))^2 + (1 - 0)^2} \\
= \sqrt{3^2 + 1^2} \\
= \sqrt{9 + 1} \\
= \sqrt{10}
\]

\[
BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
= \sqrt{(2 - 1)^2 + (1 - (-2))^2} \\
= \sqrt{1^2 + 3^2} \\
= \sqrt{1 + 9} \\
= \sqrt{10}
\]

\[
CD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
= \sqrt{(1 - (-2))^2 + (-2 - (-3))^2} \\
= \sqrt{3^2 + 1^2} \\
= \sqrt{9 + 1} \\
= \sqrt{10}
\]

\[
AD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
= \sqrt{(-1 - (-2))^2 + (0 - (-3))^2} \\
= \sqrt{1^2 + 3^2} \\
= \sqrt{1 + 9} \\
= \sqrt{10}
\]

\(ABCD\) is a quadrilateral with four congruent sides, so \(ABCD\) is a rhombus.

The **midpoint** of a line segment is the point that divides the segment into two equal-length segments. You can find the coordinates of the midpoint of a line segment by using the midpoint formula.

\[
\text{The Midpoint Formula}
\]

The midpoint \(M\) of the line segment with endpoints \((x_1, y_1)\) and \((x_2, y_2)\) is

\[
M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).
\]

**Example 4** Finding the Midpoint of a Segment

Find the midpoint of the line segment with the given endpoints.

(3, 5) and (7, -2)

\[ \textbf{SOLUTION} \] Use the midpoint formula. Substitute (3, 5) for \((x_1, y_1)\) and (7, -2) for \((x_2, y_2)\).

\[
M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \\
= \left( \frac{3 + 7}{2}, \frac{5 + (-2)}{2} \right) \quad \text{Substitute.} \\
= \left( \frac{10}{2}, \frac{3}{2} \right) \quad \text{Simplify.} \\
= \left( 5, \frac{3}{2} \right) \quad \text{Simplify.}
\]

The midpoint of the line segment with endpoints (3, 5) and (7, -2) is \( \left( 5, \frac{3}{2} \right) \).
Example 5  Application: Football

A coordinate plane can be used to model positions of players on a football field.

A quarterback is on the 30-yard line at (30, 10). He throws a pass to his receiver who is on the 50-yard line at (50, 40). Find the length of the pass as a radical in simplest form. Then use a calculator to estimate the length to the nearest yard.

SOLUTION  Use the distance formula to find the distance between the quarterback and his receiver. Substitute (30, 10) for \((x_1, y_1)\) and (50, 40) for \((x_2, y_2)\).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

\[
= \sqrt{(50 - 30)^2 + (40 - 10)^2}
\]

\[
= \sqrt{20^2 + 30^2}
\]

\[
= \sqrt{400 + 900}
\]

\[
= \sqrt{1300}
\]

\[
= 10\sqrt{13} \approx 36
\]

The pass is about 36 yards long.

Lesson Practice

a. Use the diagram of city streets from Example 1. What is the direct distance (in city blocks) from the corner of C St. and 2nd Ave. to the corner of D St. and 5th Ave.? Give your answer in simplest radical form. Use a calculator to approximate the answer to the nearest whole city block.

b. Find the distance between the points \((-3, -2)\) and \((4, 2)\).

c. Determine whether quadrilateral \(PQRS\) is a rhombus.

d. Find the midpoint of the line segment with endpoints \((-2, 3)\) and \((4, 7)\).

e. Football Use a coordinate plane like the one shown in Example 5. A quarterback is on the 20-yard line at \((20, 33)\). He throws a pass to his receiver, who is on the opponent’s 58-yard line at \((58, 15)\). Find the length of the pass as a radical in simplest form. Then use a calculator to estimate the length to the nearest yard.
Solve and graph the solution set.

1. \(15y < 60\)
2. \(16 < 6x + 10\) OR \(-16 > 6x - 10\)

Factor completely.

3. \(-2g^2 - 8g + 90\)
4. \(20b^2 + 21b - 5\)
5. \(-13w^2 + 38w - 25\)

Write each equation in the standard form of a quadratic function, if possible.

6. \(y - 14x = -20x^2\)
7. \(x - 5x = -2x^2 + 7\)

*8. Find the distance between \((4, -1)\) and \((7, 3)\) using the distance formula.

9. **Personal Finance** George’s credit card company offers 4% cash back on all purchases. Write and solve an inequality to determine how many charges he needs to make in one year to earn at least $100 cash back.

10. Use the table to determine if there is a positive correlation, a negative correlation, or no correlation between the data sets.

<table>
<thead>
<tr>
<th>(x)</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>12</td>
<td>25</td>
<td>40</td>
<td>51</td>
<td>61</td>
<td>75</td>
</tr>
</tbody>
</table>

11. **Multi-Step** A real number is at most 13 and at least 5.
   a. Write two separate inequalities to describe the problem.
   b. Write the two inequalities as one compound inequality.
   c. Graph the compound inequality.

12. **Carpentry** Louis is building a new rectangular room onto his house. The room has a side length of \(3 + \sqrt{15}\) feet and a width of \(4 + \sqrt{36}\) feet. What is the area of Louis’s new room?

13. **Multi-Step** The temperature in Texas has never been above 120 degrees Fahrenheit. Describe this using Celsius temperature by solving the inequality \(120 \geq \frac{9}{5}C + 32\).

14. **Write** How do changes to the value of \(c\) affect the graph of \(y = \frac{a}{x - b} + c\)?

15. A sandwich maker chooses a meat and a vegetable at random to put on a sandwich. There are three meats: turkey, ham, and chicken. There are 5 vegetables: lettuce, tomato, cucumber, onion, and peppers. Make a table of the possible outcomes.

16. **Hobbies** Kelly goes to a local store that has a monthly fee of $5 and rents games there for $1.75 a week. An online company has no monthly fee but rents games for $2.25 a week. How many games would Kelly have to rent per month for the local store to be the better deal?
17. **Measurement** One square has an area of 16 square units and another square has an area of 36 square units. A third square has an area greater than that of the smaller square and less than that of the larger square. What are the possible lengths of the sides of the third square?

18. **Multiple Choice** What is the factored form of $32x^2 - 50y^2$?
   - A $2(4x + 5y)^2$
   - B $2(4x - 5y)^2$
   - C $2(4x + 5y)(4x - 5y)$
   - D $2(16x + 25y)(16x - 25y)$

19. **Verify** Rewrite the expression $y^2 - x^2 - 8x - 41$ as a difference of two squares minus a perfect-square polynomial to show that $y^2 - x^2 - 8x - 41 = (y + 5)(y - 5) - (x + 4)^2$.

20. **Error Analysis** Two students are asked if the equation $7x^2 + 24 = y - 6x(2 - 3x^2)$ is a quadratic function. Which student is correct? Explain the error.

<table>
<thead>
<tr>
<th>Student A</th>
<th>Student B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = -18x^3 + 7x^2 + 12x + 24$</td>
<td>$y = 10x^2 + 12x + 24$</td>
</tr>
<tr>
<td>no</td>
<td>yes</td>
</tr>
</tbody>
</table>

21. **Economics** A company has developed a new product. To determine the selling price, the company uses the function $y = -55x^2 + 1500x$ to predict the profit for selling the product for $x$ dollars. Does the graph of this function open upward or downward? Explain why this might be given the context of the situation.

22. Use the Pythagorean Theorem to find the missing side length. Give the answer in simplest radical form.

23. **Geometry** A right isosceles triangle is a right triangle whose legs are equal in length.
   a. Find the length of the hypotenuse of a right isosceles triangle with leg lengths of 3.
   b. Find the length of the hypotenuse of a right isosceles triangle with leg lengths of 5.
   c. **Formulate** Use the results of parts a and b to suggest a formula for the hypotenuse of a right isosceles triangle with legs of length $a$.

24. **Error Analysis** Two students use the Pythagorean Theorem to find length $p$. Which student is correct? Explain the error.

<table>
<thead>
<tr>
<th>Student A</th>
<th>Student B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^2 + 7^2 = p^2$</td>
<td>$2^2 + p^2 = 7^2$</td>
</tr>
<tr>
<td>$4 + 49 = p^2$</td>
<td>$4 + p^2 = 49$</td>
</tr>
<tr>
<td>$53 = p^2$</td>
<td>$p^2 = 45$</td>
</tr>
<tr>
<td>$\sqrt{53} = p$</td>
<td>$p = \sqrt{45}$</td>
</tr>
<tr>
<td></td>
<td>$p = 3\sqrt{5}$</td>
</tr>
</tbody>
</table>
25. Use the diagram of city streets from Example 1 on page 563. What is the direct distance (in city blocks) from the corner of A St. and 4th Ave. to the corner of E St. and 2nd Ave.? Give your answer in simplest radical form and to the nearest tenth of a city block.

26. Multi-Step Marisol is flying a kite as shown in the diagram.
   a. Use the Pythagorean Theorem to find the length $h$.
   b. How high is the kite off of the ground?

27. Write To find the distance between $(5, 3)$ and $(-2, 9)$, Dan lets $(x_1, y_1) = (5, 3)$ in the distance formula. Dawn lets $(x_1, y_1) = (-2, 9)$. Explain why Dan and Dawn will get the same result.

28. Justify Is the triangle with vertices at $(-3, 3)$, $(1, 0)$, and $(4, 4)$ a right triangle? Justify your answer.

29. Multiple Choice Which points are not endpoints of a line segment with midpoint $(7, -3)$?
   A $(1, 2)$ and $(13, -8)$
   B $(5, 0)$ and $(9, -5)$
   C $(2, -9)$ and $(12, 3)$
   D $(4, -2)$ and $(10, -4)$

30. Baseball A baseball diamond is a square that is 90 feet long on each side. Use a coordinate grid to model positions of players on the field; place home plate at $(0, 0)$, first base at $(90, 0)$, second base at $(90, 90)$, and third base at $(0, 90)$. An outfielder located at $(50, 300)$ throws to the third-baseman. How long is the throw? Round your answer to the nearest foot.
LESSON 87

Factoring Polynomials by Grouping

Warm Up

1. **Vocabulary** For the terms in a polynomial, the product of the greatest integer that divides evenly into the coefficients and the greatest power of each variable that divides evenly into each term is the __________.

Factor each polynomial completely.

2. \(90k^4 + 15k^3\) \(\text{Answer: } 15k^3(6k + 1)\)
3. \(x^2 - 8x + 15\) \(\text{Answer: } (x - 3)(x - 5)\)
4. \(4n^2 + 5n - 21\) \(\text{Answer: } (2n + 7)(2n - 3)\)
5. \(81x^3 - 64y^3\) \(\text{Answer: } (3x - 4y)(x^2 + 4xy + 16y^2)\)

New Concepts

Polynomials can be factored by grouping. When a polynomial has four terms, make two groups and factor out the greatest common factor from each group.

**Example 1** Factoring Four-Term Polynomials

Factor \(2x^2 + 4xy + 7x + 14y\). Check your answer.

**SOLUTION**

\[2x^2 + 4xy + 7x + 14y = (2x^2 + 4xy) + (7x + 14y)\]
Group terms that have a common factor.

\[= 2x(x + 2y) + 7(x + 2y)\]
Factor out the GCF of each binomial.

\[= (x + 2y)(2x + 7)\]
Factor out \((x + 2y)\).

**Check**

\[(x + 2y)(2x + 7) = 2x^2 + 7x + 4xy + 14y\]
Multiply using FOIL.

\[= 2x^2 + 4xy + 7x + 14y\]
Commutative Property

The product is the original polynomial.

**Example 2** Rearranging before Grouping

Factor \(3y^2 - 8y^3 - 8y + 3\). Check your answer.

**SOLUTION**

Use the Commutative and Associative Properties to rearrange terms to form two binomials with common factors.

\[3y^2 - 8y^3 - 8y + 3 = 3y^2 + 3 - 8y^3 - 8y\]
Group terms that have a common factor.

\[= (3y^2 + 3) - (8y^3 + 8y)\]
Group into two binomials.

\[= 3(y^2 + 1) - 8y(y^2 + 1)\]
Factor out the GCF of each binomial.

\[= (y^2 + 1)(3 - 8y)\]
Factor out \((y^2 + 1)\).
Check

\((y^2 + 1)(3 - 8y)\)
\[= 3y^2 - 8y^3 + 3 - 8y\]  
Multiply using FOIL.
\[= 3y^2 - 8y^3 - 8y + 3\]  
Commutative Property

The product is the original polynomial.

**Example 3**  Factoring with the Greatest Common Factor

Factor \(45a^3b - 15a^3 + 15a^2b - 5a^3\). Check your answer.

**SOLUTION**

\[= 45a^3b - 15a^3 + 15a^2b - 5a^3\]
\[= 5a^3(9ab - 3a + 3b - 1)\]  
Factor out the GCF.
\[= 5a^3[(9ab - 3a) + (3b - 1)]\]  
Group into two binomials.
\[= 5a^3[(3a)(3b - 1) + 1(3b - 1)]\]  
Factor out the GCF of each binomial.
\[= 5a^3[(3b - 1)(3a + 1)]\]  
Factor out \((3b - 1)\).

Check

\[5a^3[(3b - 1)(3a + 1)]\]
\[= 5a^3[9ab + 3b - 3a - 1]\]  
Multiply using FOIL.
\[= 45a^3b + 15a^2b - 15a^3 - 5a^3\]  
Distributive Property
\[= 45a^3b - 15a^3 + 15a^2b - 5a^3\]  
Commutative Property

The product is the original polynomial.

**Example 4**  Factoring with Opposites

Factor \(3a^2b - 18a + 30 - 5ab\) completely. Check your answer.

**SOLUTION**

\[3a^2b - 18a + 30 - 5ab\]
\[= (3a^2b - 18a) + (30 - 5ab)\]  
Group into two binomials.
\[= 3a(ab - 6) + 5(6 - ab)\]  
Factor the GCF from each binomial.
\[= 3a(ab - 6) + 5(-1)(ab - 6)\]  
Take the opposite by multiplying by \(-1\).
\[= 3a(ab - 6) - 5(ab - 6)\]  
Simplify.
\[= (ab - 6)(3a - 5)\]  
Factor out \((ab - 6)\).

Check

\[(ab - 6)(3a - 5)\]
\[= 3a^2b - 5ab - 18a + 30\]  
Multiply using FOIL.
\[= 3a^2b - 18a + 30 - 5ab\]  
Commutative Property

The product is the original polynomial.
A trinomial of the form $ax^2 + bx + c$ can also be factored by grouping. The trinomial is expressed as a polynomial with four terms so that it can be factored by grouping.

To express trinomials in the form $ax^2 + bx + c$ with four terms, first identify $a$, $b$, and $c$. For example, in the trinomial $2x^2 + 11x + 15$, $a = 2$, $b = 11$, and $c = 15$. Then, to factor a trinomial such as $2x^2 + 11x + 15$ by grouping, use the steps shown below.

**Step 1:** Find the product of $ac$.

$$2 \cdot 15 = 30$$

**Step 2:** Find two factors of $ac$ with a sum equal to $b$.

$$6 \cdot 5 = 30 \text{ and } 6 + 5 = 11$$

**Step 3:** Write the trinomial using the sum. Replace $11x$ with $6x + 5x$.

$$2x^2 + 11x + 15 = 2x^2 + 6x + 5x + 15$$

**Step 4:** Factor by grouping.

$$2x^2 + 11x + 15 = 2x^2 + 6x + 5x + 15 = (2x^2 + 6x) + (5x + 15) = 2x(x + 3) + 5(x + 3) = (x + 3)(2x + 5)$$

**Example 5  Factoring a Trinomial**

Factor each trinomial by grouping.

**a.** $x^2 - 7x - 44$

**SOLUTION**

$$ac = 1 \cdot -44 = -44; \text{ Factors of } -44 \text{ with a sum of } -7 \text{ are } -11 \text{ and } 4.$$ 

$$x^2 - 7x - 44 = x^2 - 11x + 4x - 44 \quad \text{Replace } -7x \text{ with } -11x \text{ and } 4.$$ 

$$= x(x - 11) + 4(x - 11) \quad \text{Group into two binomials.}$$ 

$$= (x - 11)(x + 4) \quad \text{Factor out the GCF of each binomial.}$$

**b.** $6k^2 - 17k + 10$

**SOLUTION**

$$ac = 6 \cdot 10 = 60; \text{ Factors of } 60 \text{ with a sum of } -17 \text{ are } -5 \text{ and } -12.$$ 

$$6k^2 - 17k + 10 = 6k^2 - 12k - 5k + 10 \quad \text{Replace } -17k \text{ with } -12k \text{ and } -5k.$$ 

$$= (6k^2 - 12k) - (5k - 10) \quad \text{Group into two binomials.}$$ 

$$= 6k(k - 2) - 5(k - 2) \quad \text{Factor out the GCF of each binomial.}$$ 

$$= (k - 2)(6k - 5) \quad \text{Factor out } (k - 2).$$

**Caution**

Remember to change the signs of terms within the parentheses when factoring out negative 1.
Lesson Practice

Factor completely. Check your answer.

a. \(3y^2 + 6yz + 4y + 8z\)
   (Ex 1)
b. \(3y^2 - 4y^3 + 3 - 4y\)
   (Ex 2)
c. \(9x^3y - 33x^3 + 33x^2y - 11x^2\)
   (Ex 3)
d. \(3a^2b - 4ab + 20 - 15a\)
   (Ex 4)

Factor each trinomial by grouping.

e. \(x^2 - 4x - 77\)
   (Ex 5)
f. \(6a^2 - 1a - 15\)
   (Ex 5)

Practice  Distributed and Integrated

Factor.

1. \(x^2 + 3xy - 54y^2\)
   (Ex 1)

2. \(64a^2b - 16a^3 + 18a^2b - 9\)
   (Ex 2)

Solve.

3. \(2g + 9 - 4g < 5 + 6g - 2\)
   (Ex 3)

4. \(6(k - 5) > 3k - 26\)
   (Ex 4)

Use an equation to find the probability of the event.

5. rolling 4 on two number cubes and a coin landing on heads
   (Ex 5)

6. rolling a number less than 4 on two number cubes and a coin landing on heads
   (Ex 6)

Determine whether the polynomial is a perfect-square trinomial or a difference of two squares. Then factor the polynomial.

7. \(100 - c^6\)
   (Ex 7)

8. \(4x^2 + 20x + 25\)
   (Ex 8)

Find the distance between the given points. Give the answer in simplest radical form.

9. \((1, 3)\) and \((4, 7)\)
   (Ex 9)

10. \((2, -1)\) and \((6, 3)\)
    (Ex 10)

11. Which situation would most likely be represented by a negative correlation: hours of practice and your golf score, hours of practice and the number of baskets you make in basketball, or hours of practice and the cost of a computer? Hint: In golf, the lower the score, the better.

12. Entertainment The table shows the average ticket price for a movie. Make a scatter plot from the data.

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<thead>
<tr>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Ticket Price (dollars)</td>
<td>4.23</td>
<td>4.35</td>
<td>5.40</td>
<td>5.65</td>
<td>5.80</td>
<td>6.03</td>
<td>6.21</td>
<td>6.41</td>
</tr>
</tbody>
</table>
13. **Multi-Step** A grower ships fruit to a processing plant in 50-pound cases. The plant will not accept cases that differ from this weight by more than ±0.5 pound.

a. Write an absolute-value equation to find the minimum and maximum weights of the cases that the processing plant will accept.

b. What are the maximum and minimum acceptable weights?

14. **Hiking** Heidi can hike the mountains at 2 miles per hour. She has already hiked 5 miles and wants to be sure to turn around before she hikes more than 9 miles. To find the number of hours she can hike before she needs to turn around, solve the inequality \(2m + 5 \leq 9\).

15. **Multi-Step** The local public library has a budget of $3000 to buy new children’s books. The library will receive 100 free books when it places its order. The number of books, \(y\), that they can get is given by \(y = \frac{3000}{x} + 100\), where \(x\) is the price per book.

a. What is the horizontal asymptote of this rational function?

b. What is the vertical asymptote?

c. If the price per book is $20, how many books will the library receive?

16. **Generalize** How do you know when a trinomial is factored completely?

17. **Painting Job** Marco paints walls and charges a $20 set-up fee. He charges at least $60 per big wall, and for small walls he charges no more than $40 per wall. He has just completed a job painting either all big walls or all small walls. His invoice states that he received $2420 in payment. How many walls could he have painted?

18. **Measurement** A map has a scale 1 cm:500 m. A circular pond on the map has an area of \(9x^2\pi - 6x\pi + \pi\) square centimeters. What is the actual diameter of the pond?

19. **Multiple Choice** The graph of which function opens downward?

A. \(-8y + 3x^2 = 4 + 7x\)

B. \(-12x^2 + 15y = 18\)

C. \(-y + 36x = x^2 + 40\)

D. \(-15 + 9y = 45x^2 - 3x\)

20. **Write** Explain how to graph a quadratic function such as \(y + 28x - 3 = 50x^2 + 7\).

21. **Art** Use the Pythagorean Theorem to find the missing side length. Give the answer in simplest radical form.

*22. **Art** Do the lengths 10, \(5\sqrt{5}\), and 15 form a right triangle?

23. **Art** Theresa is painting on a triangular canvas. The lengths of the sides of the canvas are 24 inches, 32 inches, and 42 inches. Is her canvas a right triangle?
24. **Error Analysis** Two students use the distance formula to find the length of the line segment with endpoints \((-1, 6)\) and \((4, -2)\). Which student is correct? Explain the error.

<table>
<thead>
<tr>
<th>Student A</th>
<th>Student B</th>
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</thead>
<tbody>
<tr>
<td>(d = \sqrt{(4 + 1)^2 + (-2 - 6)^2})</td>
<td>(d = \sqrt{(4 - 1)^2 + (-2 - 6)^2})</td>
</tr>
<tr>
<td>(d = \sqrt{25 + 64})</td>
<td>(d = \sqrt{9 + 64})</td>
</tr>
<tr>
<td>(d = \sqrt{89})</td>
<td>(d = \sqrt{73})</td>
</tr>
</tbody>
</table>

25. **Geometry** A midsegment of a triangle is a line segment joining the midpoints of two sides of the triangle. A triangle has vertices at \(P = (3, 2)\), \(Q = (3, 8)\), and \(R = (7, 6)\).

a. Find the endpoints \(M\) and \(N\) of the midsegment that joins sides \(PQ\) and \(QR\).

b. **Verify** Show that the length of the midsegment \(MN\) is half the length of \(PR\).

26. **Multi-Step** Use a coordinate plane to model the following situation in a football game: A quarterback is on the 25 yard line at \((25, 10)\); he has two receivers, one at \((30, 40)\) and the other at \((50, 20)\).

a. Find the quarterback’s distance to the receiver at \((30, 40)\). Give your answer to the nearest yard.

b. Find the quarterback’s distance to the receiver at \((50, 20)\). Give your answer to the nearest yard.

c. Which receiver is closer to the quarterback?

27. **Multi-Step** The area of a right triangle is \(x^2 + 2x\). The base, \(b\), equals \(x + 2\). What is the length of the height, \(h\)?

a. Write the formula for the area of a triangle given the base and height.

b. Set the area equal to the product of the base and the height.

c. Factor.

d. Divide by the length of the base to find the length of the height.

28. **Write** Is a binomial also a polynomial? Explain.

29. **Formulate** Write an expression to show the price of 2 discounted books when the first one is \(20 + n\) and each one after that is \((20 + n)(n - 5)\).

30. **Cost** The baseball team gets new uniforms. The first group of 5 costs \(y^2 + 5\) dollars, and after that, each group of 5 costs \(y + 1\) dollars. How much do the uniforms cost if the team buys 15?
Warm Up

1. **Vocabulary** A _________ (radical, rational) expression has at least one variable in its denominator.

Simplify. Assume that no denominator is equal to zero.

2. \( x^6x^2 \)

3. \( \frac{8b^4}{12b^9} \)

4. \( \frac{12y^4 - 18y^3}{28y^3 - 42y} \)

5. Factor \( x^2 - 14x + 24 \).

New Concepts

Multiplying and dividing rational expressions follows the same procedure as multiplying and dividing fractions.

<table>
<thead>
<tr>
<th>Multiplying Rational Expressions</th>
<th>Dividing Rational Expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( a, b, c, ) and ( d ) are nonzero polynomials, ( \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} ).</td>
<td>If ( a, b, c, ) and ( d ) are nonzero polynomials, ( \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} ).</td>
</tr>
</tbody>
</table>

**Example 1** Multiplying Rational Expressions

Find each product.

\[ \frac{6x^4y}{21xy^3} \cdot \frac{7x^3y^2}{3x^3y^2} \]

**SOLUTION**

\[ \frac{6x^4y}{21xy^3} \cdot \frac{7x^3y^2}{3x^3y^2} = \frac{42x^4y^3}{63x^4y^5} = \frac{2x^2}{3y^2} \]

Multiply the numerators and denominators.

\[ \frac{6x^2}{5y^4} \cdot \frac{3x}{7y^2} \]

**SOLUTION**

\[ \frac{6x^2}{5y^4} \cdot \frac{3x}{7y^2} = \frac{18x^3}{35y^6} \]

Multiply the numerators and denominators.

There are no common factors, so the product does not simplify any further.
Example 2  Multiplying a Rational Expression by a Polynomial

Multiply \( \frac{9}{3x - 15} \cdot (x^2 - 2x - 15) \). Simplify your answer.

**SOLUTION**

\[
\frac{9}{3x - 15} \cdot (x^2 - 2x - 15)
\]

\[
= \frac{9}{3x - 15} \cdot \frac{(x^2 - 2x - 15)}{1}
\]

\[
= \frac{9}{3(x - 5)} \cdot \frac{(x - 5)(x + 3)}{1}
\]

\[
= \frac{3}{1} \cdot \frac{(x - 5)(x + 3)}{1}
\]

\[
= 3(x + 3)
\]

\[
= 3x + 9
\]

There is more than one way to find the product of two rational expressions. The factors can be multiplied first before being simplified. Another way to solve the problem would be to simplify each expression first, and then to multiply, simplifying again if necessary.

Example 3  Multiplying Rational Expressions Containing Polynomials

Multiply \( \frac{8m^2n + 2mn}{2m} \cdot \frac{15}{24mn + 6n} \). Simplify your answer.

**SOLUTION**

**Method 1:** Multiply first.

\[
\frac{8m^2n + 2mn}{2m} \cdot \frac{15}{24mn + 6n}
\]

\[
= \frac{15(8m^2n + 2mn)}{2m(24mn + 6n)}
\]

\[
= \frac{120m^2n + 30mn}{48mn + 12mn}
\]

\[
= \frac{30mn(4m + 1)}{12mn(4m + 1)}
\]

\[
= \frac{5}{2}
\]

**Method 2:** Factor first.

\[
\frac{8m^2n + 2mn}{2m} \cdot \frac{15}{24mn + 6n}
\]

\[
= \frac{2mn(4m + 1)}{2m} \cdot \frac{15}{6n(4m + 1)}
\]

\[
= \frac{5}{2}
\]
Example 4 Dividing Rational Expressions

Find each quotient.

a. \( \frac{5st^4}{4s^2t} \div \frac{15s^2t}{2s^3t^2} \)

SOLUTION
\[
\frac{5st^4}{4s^2t} \div \frac{15s^2t}{2s^3t^2} = \frac{5st^4}{4s^2t} \cdot \frac{2s^3t^2}{15s^2t} \]
Write as multiplication by the reciprocal.
\[
= \frac{10s^4t^6}{60s^4t^2} \]
Multiply the numerators and denominators.
\[
= \frac{r^6}{6} \]
Simplify.

b. \( \frac{9r^2 - 12r}{27} \div (3r - 4) \)

SOLUTION
\[
\frac{9r^2 - 12r}{27} \div (3r - 4) = \frac{9r^2 - 12r}{27} \div \frac{(3r - 4)}{1} \]
Write the polynomial with a denominator of 1.
\[
= \frac{9r^2 - 12r}{27} \cdot \frac{1}{(3r - 4)} \]
Write as multiplication by the reciprocal.
\[
= \frac{3r(3r - 4)}{27} \cdot \frac{1}{(3r - 4)} \]
Factor.
\[
= \frac{1}{9} \cdot \frac{3r(3r - 4)}{(3r - 4)} \]
Divide out like factors.
\[
= \frac{r}{9} \]
Simplify.

c. \( \frac{x^2 + 4x + 3}{x^2} \div \frac{x + 3}{x} \)

SOLUTION
\[
\frac{x^2 + 4x + 3}{x^2} \div \frac{x + 3}{x} = \frac{x^2 + 4x + 3}{x^2} \cdot \frac{x}{x + 3} \]
Write as multiplication by the reciprocal.
\[
= \frac{(x + 1)(x + 3)}{x^2} \cdot \frac{x}{(x + 3)} \]
Factor. Divide out like factors.
\[
= \frac{x + 1}{x} \]
Simplify.
**Example 5** Application: Profit

A business makes a profit of \( \frac{x^4}{100x^2 + 100x} \) dollars for each item sold. If \( x^2 + 5x + 4 \) items are sold, what is the total profit in terms of \( x \)?

**SOLUTION**

Multiply the profit for each item sold by the amount of items sold.

\[
\frac{x^4}{100x^2 + 100x} \cdot (x^2 + 5x + 4)
\]

\[
= \frac{x^4}{100x(x + 1)} \cdot \frac{x^2 + 5x + 4}{1}
\]

Write the polynomial with a denominator of 1.

\[
= \frac{x^4}{100x(x + 1)} \cdot \frac{(x + 4)(x + 1)}{1}
\]

Factor.

\[
= \frac{x^4}{100x(x + 1)} \cdot \frac{(x + 4)(x + 1)}{1}
\]

Divide out like factors.

\[
= \frac{x^3(x + 4)}{100}
\]

Simplify.

**Lesson Practice**

Find each product.

(a) \( \frac{4z^3q^8}{14qz^7} \cdot \frac{14qz^4}{3q^2z} \)

(b) \( \frac{5x^2}{7y^4} \cdot \frac{4x^2}{9y^3} \)

Multiply. Simplify your answer.

(c) \( \frac{6}{2x - 18} \cdot (x^2 - 6x - 27) \)

(d) \( \frac{8m + 6m^2n}{12} \cdot \frac{8m}{24m + 8mn} \)

Find each quotient.

(e) \( \frac{8j^2k^3}{15k^2j^4} \div \frac{6j^3k}{5kj^6} \)

(f) \( \frac{x^2 + 7x + 12}{x + 5} \div (x + 3) \)

(g) \( \frac{x^2 + 5x + 6}{x + 2} \div \frac{x + 3}{y^2} \)

(h) Profits Tran makes a profit of \( \frac{x^4}{20x^2 + 10x} \) for each ticket he sells. What is his profit in terms of \( x \) if he sells \( x^2 + 9x + 20 \) tickets?
1. Solve \(6(x + 2) - 4x > 2 + x + 2\).

2. Solve and graph the inequality \(4 \geq 2(x + 3) \text{ OR } 23 < 8x + 7\).

Rewrite the equation in the standard form of a quadratic function, if possible.

3. \(x(y - 2x) = 18x^2\)

4. \(2(y - 2x) = 6x^2\)

Find the distance between the given points. Give the answer in simplest radical form.

5. \((-4, -5) \text{ and } (2, -3)\)

6. \((3, -2) \text{ and } (1, 0)\)

Find the midpoint of the line segment with the given endpoints.

7. \((-4, -5) \text{ and } (2, -3)\)

8. \((3, -2) \text{ and } (1, 0)\)

Find the product or quotient.

*9. \(\frac{2y^2 + 10y}{y + 5} \cdot \frac{2x}{2x^3}\)

*10. \(\frac{6y}{x^2} \div \frac{x + y}{3y}\)

*11. \(\frac{7x}{y} \div \frac{4}{y}\)

Factor.

12. \(7x - 60 + x^2\)

13. \(64a^3b - 32a^3 + 16a^2b - 8a^2\)

14. (Woodworking) To inlay designs on the top of chest, a crafter uses a series of patterns. The patterns are represented by the trinomial \(x^2 + 4x - 21\). What are the dimensions of the patterns?

15. Write a compound inequality that describes the graph.

16. (Athletic Directing) An athletic director has a budget of $700 to buy uniforms. He will receive 5 free uniforms when he places his order. The number of uniforms, \(y\), that he can get is given by \(y = \frac{700}{x} + 5\), where \(x\) is the price per uniform.

a. What is the horizontal asymptote of this rational function?

b. What is the vertical asymptote?

c. If the price per uniform is $50, how many uniforms will he receive?

d. Can the athletic director receive only 5 uniforms? Why or why not?

17. (Multi-Step) The area of a triangular sail in square feet is represented by the equation \(\frac{1}{2}x^2 + \frac{7}{2}x + 5\).

a. Factor this expression completely in terms of the area of a triangle.

b. Let \(x = 3\). What is the area of the sail in square feet?
18. Write Why is making tables and graphs a good way to show probability distribution?

19. Paving A new restaurant is opening in a square building with a side length of \( s \) feet. A parking lot surrounds the building in the form of a larger square. The restaurant sits in the middle of the lot. The plot of land on which the restaurant and parking lot lie has an area of \( 9s^2 + 54s + 81 \) square feet. How far does the parking lot extend from the building?

*20. Analyze* The figure shows the first four right triangles in the *Wheel of Theodorus* (named after a fifth-century Greek philosopher).

![Diagram of triangles]

a. Find the length of the hypotenuse of each of the four triangles.

b. **Predict** Suppose the pattern of the triangles is continued until the figure contains 10 triangles. Predict what the length of the hypotenuse of the tenth triangle will be.

21. Multiple Choice Which lengths represent the lengths of the sides of a right triangle?

A 3, 4, 6

B \( \sqrt{13} \), 5, 12

C \( \sqrt{15} \), 7, 8

D 6, 8, 12

22. Error Analysis Students A and B find the midpoint of the line segment with endpoints \( P = (12, -5) \) and \( Q = (-2, 3) \). Student A says that the \( x \)-coordinate of the midpoint is \( \frac{12 + (-2)}{2} = \frac{10}{2} = 5 \). Student B says that the \( x \)-coordinate of the midpoint is \( \frac{12 - (-2)}{2} = \frac{14}{2} = 7 \). Which student is correct? Explain the error.

23. Software A game designer is working on a computer basketball game. On the computer screen, the coordinates of the corners of the court are (20, 20), (20, 70), (114, 70), and (114, 20). Find the length of a diagonal of the court to the nearest tenth of a unit.
24. **Measurement**  Malik is adding 5 feet to the length of a screened-in porch. If the porch is originally a square room with area \( x^2 \) and the new area measures 50 square feet, what are the dimensions of the new porch?

25. **Geometry**  A rectangle has a length of \( 12x^2 + 3y \) and a width of \( 6x^2 + y \). What is its area? What is the simplest way to express the area?

26. **Multiple Choice**  What is the result of a complete factoring of \( 25x^2 - 81 \)?
   
   A  \( 25(x + 9)(x - 9) \)  
   B  \( 5x \cdot 5x - 81 \)  
   C  \( 5^2x^2 - 9^2 \)  
   D  \( (5x + 9)(5x - 9) \)

27. **Generalize**  What is the method for multiplying and dividing exponents when working with rational expressions?

28. **Multiple Choice**  Multiply \( \frac{y^2 + 6y + 5}{y^2} \cdot \frac{y}{y + 1} \).
   
   A  \( \frac{y + 5}{y^2} \)  
   B  \( \frac{y + 5}{y} \)  
   C  \( \frac{(y + 1)(y + 5)}{y^2} \)  
   D  \( y(y + 5) \)

29. **Murals**  Lucy paints murals. She charges $10.00 per square foot. If she paints a mural that is \( c^2 + 7c + 10 \) ft by \( \frac{1}{c + 5} \) ft, how much does Lucy charge for painting the mural?

30. **Multi-Step**  The area of a square patio is represented by \( 81x^2 - 36x + 4 \) square feet.
   
   a. Factor the expression completely.
   
   b. Why do you think this expression is called a perfect square?
A zero of a function is an $x$-value where $f(x) = 0$. You can find the zeros of a parabola using a hand-drawn graph or algebraically using the equation. You can use a graphing calculator to compute approximate values of any $x$-intercepts and the maximum or minimum of the parabola.

Find any $x$-intercepts and the maximum or minimum of the parabola $y = -2x^2 + 4x - 2$.

1. Enter the equation $y = -2x^2 + 4x - 2$ into the Y= editor.

2. Press ZOOM 6:ZStandard to graph the equation.

3. Press TRACE and then use the ▼ and ▶ keys to move along the curve to the approximate $x$-intercept. The coordinates appear at the bottom of the screen. The $x$-intercept occurs close to the point (1.064, -0.008).

4. Find more accurate coordinates of the $x$-intercept.
   Press 2nd TRACE and select 2:zero.
   Use the ▼ key to trace along the curve to a point to the left of the $x$-intercept and then press ENTER.
   Press the ▶ key to trace along the curve to a point to the right of the $x$-intercept and then press ENTER.
   Press the ▼ key to trace to a point near the $x$-intercept and then press ENTER.
   The approximate coordinates appear at the bottom of the screen. The $x$-intercept of the parabola occurs of about the point (1, 0).

For this parabola, the $x$-intercept is also the maximum of the function.
Find any $x$-intercepts and the maximum or minimum of the parabola $y = x^2 - x - 2$.

5. Enter the equation $y = x^2 - x - 2$ into the $Y=$ editor.

6. Press ZOOM 6:ZStandard to graph the equation.

7. Use the keystrokes in Step 4 to find the coordinates of the two $x$-intercepts.

The $x$-intercepts of the parabola are $(-1, 0)$ and $(2, 0)$.

8. Find the minimum of the parabola.

Press 2nd TRACE and select 3:minimum.

Use the $\text{left arrow}$ key to trace along the curve to a point on the left of the minimum and then press ENTER. Press the $\text{right arrow}$ key to trace along the curve to a point to the right of the minimum and then press ENTER. Press the $\text{up arrow}$ key to trace to a point near the minimum and then press ENTER. The approximate coordinates appear at the bottom of the screen. Decimal values may vary slightly. The minimum of the parabola occurs at about the point $(0.5, -2.25)$.

This parabola has three distinct characteristic points: two $x$-intercepts–or zeros–and one minimum.

---

**Lab Practice**

a. Graph the parabola $y = x^2 - 2x - 3$. Find any $x$-intercepts and the maximum or minimum of the parabola.

b. Graph the parabola $y = -x^2 - x + 6$. Find any $x$-intercepts and the maximum or minimum of the parabola.
Lesson 89
Identifying Characteristics of Quadratic Functions

Warm Up

1. **Vocabulary** The U-shaped graph of a quadratic function is a(n) ________ (ellipse, parabola).

For each quadratic function, tell whether the graph opens upward or downward.

2. \( y = -3x^2 + x - 11 \)

3. \( y = -8 - 7x + x^2 \)

4. Evaluate \( y = x^2 - 4x + 5 \) for \( x = -3 \).

5. **Multiple Choice** Which of these functions is represented by this graph?
   - A \( y = 2x \)
   - B \( y = -2x \)
   - C \( y = 2x^2 \)
   - D \( y = -2x^2 \)

New Concepts

The **vertex of a parabola** is the highest or lowest point on a parabola. It is the parabola’s “turning point.”

The **minimum of a function** is the least possible value of a function and the **maximum of a function** is the greatest possible value of a function. It is the \( y \)-value of the lowest or highest point on the graph of a function. On a parabola, the minimum or maximum is the \( y \)-coordinate of the vertex.

**Example 1** Identifying the Vertex and the Maximum or Minimum

Given the coordinates of each parabola’s vertex. Then give the minimum or maximum value and the domain and range of the function.

**SOLUTION**

The vertex appears to be at \((0, -4)\). It is the lowest point, so the minimum of the function is \(-4\). The domain is the set of all real numbers; the range is the set of all real numbers greater than or equal to \(-4\).

**SOLUTION**

The vertex appears to be at \((2, 5)\). It is the highest point, so the maximum of the function is \(5\). The domain is the set of all real numbers; the range is the set of all real numbers less than or equal to \(5\).
Intercepts, zeros, and roots are related. An $x$-intercept of a function is the $x$-coordinate of the point where the graph of an equation intersects the $x$-axis. A **zero of a function** is the value of $x$ that makes $f(x) = 0$, or $y = 0$. Because $y = 0$ for every value of $x$ on the $x$-axis, the zeros of a function are the same as the $x$-intercepts.

**Example 2 Finding Zeros from the Graph**

Find the zeros of each function shown in the graph.

**a.**

Math Language

The **zeros** of a quadratic function $f(x) = ax^2 + bx + c$ are the roots of the related equation $0 = ax^2 + bx + c$.

SOLUTION

The $x$-intercepts appear to be at $-1$ and $3$. Check by substituting the ordered pairs $(-1, 0)$ and $(3, 0)$ into the function.

\[
y = -x^2 + 2x + 3
\]

$0 \overleftarrow{=} -(-1)^2 + 2 \cdot -1 + 3$

$0 \overleftarrow{=} -1 - 2 + 3$

$0 = 0 \checkmark$

The zeros are $-1$ and $3$.

**b.**

SOLUTION

The $x$-intercept appears to be at $-2$. Check by substituting the ordered pair $(-2, 0)$ into the function.

\[
y = 3x^2 + 12x + 12
\]

$0 \overleftarrow{=} 3(-2)^2 + 12 \cdot -2 + 12$

$0 \overleftarrow{=} 12 - 24 + 12$

$0 = 0 \checkmark$

The zero is $-2$. 
Lesson 89

SOLUTION

The graph does not cross the \(x\)-axis, so there are no \(x\)-intercepts, and therefore no real zeros.

An axis of symmetry is a line that divides a figure or graph into two mirror-image halves. All parabolas have an axis of symmetry that passes through the vertex of the parabola.

In the figure, notice that the equation of the axis of symmetry includes the \(x\)-coordinate of the vertex, 2. Also notice that 2 is the average of the zeros, 1 and 3.

Example 3 Finding the Axis of Symmetry Using Zeros

Find the axis of symmetry for each graph.

SOLUTION

When there is one zero, the zero occurs at the vertex point, so the vertex is at \((-4.5, 0)\). Use the \(x\)-coordinate of the vertex to identify the axis of symmetry: \(x = -4.5\).

SOLUTION

Average the zeros to find the \(x\)-coordinate of the vertex:
\[
\frac{-3 + 5}{2} = \frac{2}{2} = 1.
\]
The \(x\)-coordinate of the vertex is 1. Since the axis of symmetry passes through the vertex, the equation for the axis of symmetry is \(x = 1\).
The axis of symmetry can also be found by using a formula.

### Axis of Symmetry Formula

The axis of symmetry for the graph of a quadratic equation $y = ax^2 + bx + c$ is $x = \frac{-b}{2a}$.

### Example 4 Finding the Axis of Symmetry Using the Formula

Find the axis of symmetry for the graph of each quadratic function.

**a.** $y = x^2 + 6x + 5$

**SOLUTION**

\[
x = \frac{-b}{2a} = \frac{-6}{2(1)} = -3 \quad \text{Simplify.}
\]

The equation of the axis of symmetry is $x = -3$.

**b.** $y = -2x^2 + 3x - 1$

**SOLUTION**

\[
x = \frac{-b}{2a} = \frac{-3}{2(-2)} = \frac{3}{4} \quad \text{Simplify.}
\]

The equation of the axis of symmetry is $x = \frac{3}{4}$.

### Example 5 Application: Height of a Golf Ball

A golf ball is hit from an elevated platform 10 feet above the ground. It starts with a vertical speed of 160 feet per second. Ignoring friction, the equation $y = -16t^2 + 160t + 10$ gives the height, $y$, as a function of time, $t$. Find the highest point the ball reaches and how long it takes to reach it.

**SOLUTION** Find $t$ at the vertex by using the formula for the axis of symmetry.

\[
t = \frac{-b}{2a} \quad \text{axis of symmetry formula}
\]

\[
= \frac{-160}{2(-16)} = 5 \quad \text{Use } a = -16 \text{ and } b = 160. \quad \text{Simplify.}
\]

Substitute 5 for $t$ in the equation to find the $y$-value of the vertex.

\[
y = -16t^2 + 160t + 10
\]

\[
= -16(5)^2 + 160(5) + 10 \quad \text{Substitute 5 for } t.
\]

\[
= 410 \quad \text{Simplify.}
\]

The vertex is at $(5, 410)$. This means that 5 seconds after the golf ball is hit, it reaches a maximum height of 410 feet.
Lesson Practice

Give the coordinates of each parabola’s vertex. Then give the minimum or maximum value and domain and range of the function.

(Ex 1)

a. \( y = -x^2 - 2x - 5 \)

b. \( y = x^2 - 10x + 22 \)

Find the zeros of each function.

(Ex 2)

c. \( f(x) = x^2 - 12x + 36 \)

d. \( f(x) = x^2 + 12x + 27 \)

e. \( f(x) = -x^2 - 2.5 \)

Find the axis of symmetry for each graph.

(Ex 3)

f. \( f(x) = -x^2 + 12x + 4 \)

g. \( f(x) = x^2 + 12x + 27 \)

Find the axis of symmetry for the graph of each quadratic function.

(Ex 4)

h. \( y = 3x^2 + 12x + 4 \)

i. \( y = -x^2 + 6x + 5 \)

j. A chunk of lava flies out of a volcano from a height of 12,447 feet at a velocity of 608 feet per second. Find the highest point the lava reaches and how long it takes to reach it using the equation \( y = -16t^2 + 608t + 12,447 \).
Solve each equation and check your answer.

1. \(-2|r + 2| = -30\)

2. \(3|r + 6| = 15\)

3. Solve and graph the inequality \(35 < 3x + 8\) OR \(72 \geq 9(x + 1)\).

Determine whether the polynomial is a perfect-square trinomial or a difference of two squares. Then factor the polynomial.

4. \(100y^2 - 80y + 16\)

5. \(81x^2 - 1\)

6. Factor \(9c^2 - 42c + 49\).

7. Find the distance between \((2, 2)\) and the \((4, 4)\). Give your answer in simplest radical form.

Find the quotient or product.

8. \(\frac{15a + 5b}{5ab} \div 3a + b\)

9. \(4x + 2 \div \frac{2x + 1}{3y}\)

Find the axis of symmetry for the graph of each equation.

*10. \(y = x^2 + 4x + 6\)

*11. \(y = x^2 - x - 6\)

*12. Analyze Describe how you could use symmetry to graph a quadratic equation.

13. Traffic You are driving on a highway that has a speed limit of 65 miles per hour and a minimum speed of 45 miles per hour. Write a compound inequality that describes this situation.

14. Use the graph to write a compound inequality as two separate inequalities. Then write a compound inequality without using the word AND.

15. Multi-Step Jordan is making a model of a trapezoid for her math class. Write an expression for the area of the trapezoid shown at right. Find the area.

16. Tennis A tennis ball is served at 87 miles per hour. In the expression \(-16x^2 + 128x - 240\), \(x\) represents the time the ball is 246 feet high. Factor this expression completely.

17. Multi-Step A spinner is spun 100 times. The spinner lands on brown 50 times, red 25 times, and yellow 25 times.

a. Find the experimental probability for each color.

b. Draw a spinner that would also give this theoretical probability distribution.

18. Communication A 400-anytime-minutes plan costs $35 plus $0.09 per minute overage. Another 400-anytime-minutes plan costs $45 plus $0.06 per minute overage. After how many overage minutes is the first plan more expensive?
19. **Verify** Solve the inequality $2x + 8 > 2 + 5x + 6$. Then select some values of $x$ and substitute them into the inequality to verify the solution set.

20. **Bridges** A footbridge follows the graph of the equation $y = -\frac{1}{2}x^2 + 1$. If the $x$-axis represents the ground and each unit on the graph represents 1 foot, what is the horizontal distance across the bridge?

21. **Geometry** A rectangular prism and a cylinder are both 10 inches tall. The base of the prism is a square with a side length $x$. The cylinder has a radius $x$. Graph the functions of the volumes of each solid in a coordinate plane. Then compare the graphs.

22. Use the Pythagorean Theorem to find the missing side length. Give the answer in simplest radical form.

23. **Multiple Choice** Which point is the midpoint of the line segment joining (3, 9) and (−1, 2)?
   
   A $\left(2, \frac{11}{2}\right)$  
   B $\left(2, \frac{7}{2}\right)$  
   C $\left(-2, \frac{-7}{2}\right)$  
   D $\left(1, \frac{11}{2}\right)$

24. Describe the transformation of $f(x) = -4x$ from the linear parent function.

25. **Multi-Step** A company is making various boxes to ship different-sized globes. The boxes have a volume of $V_{box} = \frac{5r^2h^2}{4} + \frac{8rh^2}{3} + \frac{2r^2h}{r}$. The globes have a volume of $V_{globe} = \frac{4\pi rh^2}{3}$. What fraction of the box do the globes take up?
   
   a. Solve for the volume of the box.
   b. Find the fraction of the box that the globes take up.

26. **Geometry** A rectangle has length $\frac{3x^2 + x}{y}$ and width $x + 2y$. What is its area?

27. **Error Analysis** Two students are asked to solve the equation $\frac{6x^2 - 3x}{15} \div 2s - 1$. Which student is correct? Explain the error.

<table>
<thead>
<tr>
<th>Student A</th>
<th>Student B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{6x^2 - 3x}{15} \div 2s - 1$</td>
<td>$\frac{6x^2 - 3x}{15} \div 2s - 1$</td>
</tr>
<tr>
<td>$\frac{6x^2 - 3x}{15} \cdot \frac{2s - 1}{1} = \frac{s(2s - 1)^2}{5}$</td>
<td>$\frac{6x^2 - 3x}{15} \cdot \frac{2s - 1}{1} = \frac{s}{5}$</td>
</tr>
</tbody>
</table>

28. **Multiple Choice** What is the $x$-coordinate of the vertex of the graph of a quadratic function whose zeros are 0 and −8?
   
   A $-4$  
   B $0$  
   C $4$  
   D $8$

29. **Space** If it were possible to play ball on Saturn, the function $y = -5.5x^2 + 44x$ would approximate the height of a ball kicked straight up at a velocity of 44 meters per second, where $x$ is time in seconds. Find the maximum height of the ball and the time it takes the ball to reach that height.

30. **Analyze** How does the value of $a$ in a quadratic equation indicate if the graph of the equation has a minimum or a maximum?
Warm Up

1. **Vocabulary** One of two or more numbers or expressions that are multiplied to get a product is a(n) _________ of the product.

2. Simplify \( \frac{7}{12} + \frac{5}{18} \).

3. Simplify \( 3x^4y + 7y^2 - 4x^4y \).

4. Multiply: \( 6r^2(3r - 8) \).

5. Multiply: \( (x - 7)(x - 3) \).

New Concepts

Adding and subtracting rational expressions follow the same rules as adding and subtracting fractions. If the denominators are the same, add the numerators and keep the common denominator. If the denominators are not the same, use the least common multiple of the denominators as the common denominator.

**Example 1** Adding and Subtracting with Like Denominators

Add or subtract. Simplify your answers.

\[ \frac{2x^2}{20x} + \frac{3x^2}{20x} \]

**SOLUTION**

\[ \frac{2x^2}{20x} + \frac{3x^2}{20x} = \frac{5x^2}{20x} \]

Add the numerators. Keep the denominator.

\[ = \frac{x}{4} \]

Simplify.

\[ \frac{3a - 2}{a + 2} - \frac{a - 6}{a + 2} \]

**SOLUTION**

\[ \frac{3a - 2}{a + 2} - \frac{a - 6}{a + 2} = \frac{3a - 2 - (a - 6)}{a + 2} \]

Subtract the numerators. Keep the denominator.

\[ = \frac{3a - 2 - a + 6}{a + 2} \]

Distribute -1.

\[ = \frac{2a + 4}{a + 2} \]

Combine like terms.

\[ = \frac{2(a + 2)}{a + 2} \]

Factor and divide out common factors.

\[ = 2 \]
Lesson 90

\[ \frac{z^3 - 10z^2}{z^2 - 3z - 18} + \frac{4z^2}{z^2 - 3z - 18} \]

**SOLUTION**

\[ \frac{z^3 - 10z^2}{z^2 - 3z - 18} + \frac{4z^2}{z^2 - 3z - 18} = \frac{z^3 - 10z^2 + 4z^2}{z^2 - 3z - 18} \]

Add the numerators. Keep the denominator.

\[ = \frac{z^3 - 6z^2}{z^2 - 3z - 18} \]

Combine like terms.

\[ = \frac{z^2(z - 6)}{(z - 6)(z + 3)} \]

Factor. Divide out common factors.

\[ = \frac{z^2}{z + 3} \]

Simplify.

When the rational expressions being added or subtracted do not have common denominators, use the least common multiple (LCM) to rename each expression.

**Example 2**  Adding with Unlike Denominators

Add. Simplify your answers.

\[ \frac{2h^4}{6h} + \frac{4h^2}{2h^2} \]

**SOLUTION**

Rename the expressions so that each has a denominator of \(6h^2\).

\[ \frac{2h^4}{6h} + \frac{4h^2}{2h^2} = \frac{2h^4}{6h} \left( \frac{h}{h} \right) + \frac{4h^2}{2h^2} \left( \frac{3}{3} \right) \]

Multiply to get common denominators.

\[ = \frac{2h^5}{6h^3} + \frac{12h^2}{6h^3} \]

Multiply. The LCD is \(6h^3\).

\[ = \frac{2h^5 + 12h^2}{6h^3} \]

Add the numerators. Keep the denominator.

\[ = \frac{2h^2(h^3 + 6)}{2h^2(3)} \]

Factor and divide out the common factors.

\[ = \frac{h^3 + 6}{3} \]

Simplify.

**Hint**

Remember to multiply the numerator and denominator by the same value when finding a common denominator.
**Example 3** Subtracting with Unlike Denominators

Subtract. Simplify your answers.

\[ \frac{x + 3}{x - 4} - \frac{2}{x^2 + x - 20} \]

**SOLUTION**

\[ \frac{x + 3}{x - 4} - \frac{2}{x^2 + x - 20} \]

Factor the denominators.

\[ = \frac{x + 3}{x - 4} - \frac{2}{(x - 4)(x + 5)} \]

Write each expression using the LCD.

\[ = \frac{x + 3(x + 5)}{(x - 4)(x + 5)} - \frac{2}{(x - 4)(x + 5)} \]

Multiply.

\[ = \frac{x^2 + 8x + 15}{(x - 4)(x + 5)} - \frac{2}{(x - 4)(x + 5)} \]

Subtract.

The numerator cannot be factored; the expression is in simplified form.

**Example 4** Simplifying with Opposite Denominators

Add. Simplify your answer.

\[ \frac{8}{v - 4} + \frac{v - 7}{4 - v} \]

**SOLUTION**

The denominators are opposites. Multiplying either denominator by \(-1\) will make both denominators the same.

\[ \frac{8}{v - 4} + \frac{v - 7}{4 - v} \]

Multiply the numerator and denominator by \(-1\).

\[ = \frac{8}{v - 4} \left( \frac{-1}{-1} \right) + \frac{v - 7}{4 - v} \]

Simplify.

\[ = \frac{-8}{4 - v} + \frac{v - 7}{4 - v} \]

Add numerators.

\[ = \frac{-8 + v - 7}{4 - v} \]

Combine like terms.

\[ = \frac{v - 15}{4 - v} \]

**Example 5** Application: Transportation

A plane flies 2000 miles with a headwind and makes the return trip with a tailwind. Write and simplify an expression for the total time of the round-trip flight assuming that the wind speed \(w\) remains constant and the plane’s rate averages 500 miles per hour.
**SOLUTION**

Rearrange the distance formula \( t = \frac{d}{r} \), and use it to write expressions for the time of each flight. Then add the expressions.

\[
\text{time going } = \frac{2000}{500 - w} \quad \text{time returning } = \frac{2000}{500 + w}
\]

\[
\text{total time } = \frac{2000}{500 - w} + \frac{2000}{500 + w}
\]

\[
= \frac{2000}{500 - w} \left( \frac{500 + w}{500 + w} \right) + \frac{2000}{500 + w} \left( \frac{500 - w}{500 - w} \right)
\]

\[
= \frac{1,000,000 + 2000w}{(500 - w)(500 + w)} + \frac{1,000,000 - 2000w}{(500 - w)(500 + w)}
\]

\[
= \frac{2,000,000 + 2000w - 2000w}{(500 - w)(500 + w)} = \frac{2,000,000}{(500 - w)(500 + w)}
\]

The expression \( \frac{2,000,000}{(500 - w)(500 + w)} \), where \( w \) is the wind speed, represents the time of the round-trip flight.

---

**Math Reasoning**

**Estimate** About how long is the round-trip flight if the wind speed is 50 mph?

---

**Lesson Practice**

Add or subtract. Simplify your answers.

\[
a. \quad \frac{4mn}{24m} + \frac{11mn}{24m} \quad (\text{Ex 1})
\]

\[
b. \quad \frac{7y - 2}{y + 6} - \frac{y - 38}{y + 6} \quad (\text{Ex 1})
\]

\[
c. \quad \frac{d^4 + 2d^3}{d^2 - 5d - 36} + \frac{2d^3}{d^2 - 5d - 36} \quad (\text{Ex 1})
\]

\[
d. \quad \frac{-3p + 2p^3}{6p^2} \quad (\text{Ex 2})
\]

\[
e. \quad \frac{x}{x + 3} - \frac{3}{x^2 + 5x + 6} \quad (\text{Ex 3})
\]

\[
f. \quad \frac{-1}{t^3 - 2} + \frac{t + 9}{2 - t^4} \quad (\text{Ex 5})
\]

\[g. \quad \text{A kayaker paddles 5 miles one way against the current and then makes the return trip with the current. Write and simplify an expression for the total time of the kayaking trip, assuming that the rate of the current remains constant and that the kayaker’s paddling rate averages 1.5 miles per hour.} \]

---

**Practice**

Find the distance between the given points. Give the answer in simplest radical form.

\[1. \quad \text{(-3, -1) and (4, 2)} \quad (\text{Ex 1})
\]

\[2. \quad \text{(1, 1) and (9, 1)} \quad (\text{Ex 1})
\]

Factor completely.

\[3. \quad 12 + 17x + 6x^2 \quad (\text{Ex 2})
\]

\[4. \quad 21t + 4t^2 - 49 \quad (\text{Ex 1})
\]

Determine whether the polynomial is a perfect-square trinomial or a difference of two squares. Then factor the polynomial.

\[5. \quad 9x^4 + 42x^3y + 49y^2 \quad (\text{Ex 1})
\]

\[6. \quad x^6 + 16x^3 + 64 \quad (\text{Ex 1})
\]
7. Graph the function \( y = \frac{-x^2}{2} \).

8. Multiply \( \frac{2x - 16}{6x^2} \cdot \frac{3y^2 + 3x}{3x - 24} \).

9. Solve \( |n + 3| + 4 = 4 \).

10. Find the axis of symmetry for the graph of the equation \( y = -3x^2 + 8x + 1 \).

Add or subtract.

\[
\begin{align*}
\text{11.} & \quad \frac{7x}{y} - \frac{2}{y} \\
\text{12.} & \quad \frac{4y}{x} - \frac{5y}{2x}
\end{align*}
\]

13. **Marine Biology** A marine biologist netted a sample of wild sardines from the northern Pacific. In his analysis of the sample, the average length of an adult male fish was 210 millimeters. He noted that the greatest variation in length from the average length in the sample was \( \pm 33 \) millimeters. Write and solve an absolute-value equation to find the lengths of the longest and shortest sardines in the sample.

14. **Multi-Step** A student has grades 75 and 90 on his first two math tests. He wants to have an average of at least 80 after his third test. To find what score he needs, solve the inequality \( \frac{75 + 90 + x}{3} \geq 80 \).

15. **Chess** All the chess pieces are put in a bag and one is drawn randomly. There are 2 kings, 2 queens, 4 rooks, 4 bishops, 4 knights, and 16 pawns. Make a graph to show the probability distribution of the pieces.

16. **Multi-Step** Marisa is changing her dog’s food from a puppy formula to an adult formula. The vet recommended that she exchange 2 ounces of adult formula for 3 ounces of puppy formula each day. Her dog usually eats 20 ounces of food per day. After how many days will the dog’s daily diet include more adult formula than puppy formula?
   a. Write and solve an inequality.
   b. Explain the correct domain of the solution set.

17. **Write** What does an AND inequality mean?

18. **Coordinate Geometry** Use the points \((2, 2), (5, 2),\) and \((2, -3)\).
   a. Write Explain why the triangle is a right triangle.
   b. What are the lengths of the legs of the triangle?
   c. Use the Pythagorean Theorem to find the length of the hypotenuse.

19. **Astronomy** Eli is looking at three stars on a star map. He measures and records the distances between each pair of stars: 9 centimeters, 12 centimeters, and 15 centimeters. Do the stars on the map determine a right triangle?

20. **Multi-Step** The area of a square is equal to \( x^2 + 6x + 9 \). What is the side length of the square?
   a. Write the formula for the area of a square.
   b. Factor the area.
   c. Set the expressions equal to each other.
   d. Find the square root of each side.
21. The area of a triangle varies jointly with the base and height of the triangle. Given that the area of a triangle is 6 square feet, the base is 3 feet, and the height is 4 feet, find the constant of variation. Express the area of a triangle.

22. **Multiple Choice** Divide \( \frac{3mn^2}{4m^2n} \div \frac{9mn}{8m^2n^2} \).

   A. \( \frac{2mn^2}{3m} \)  
   B. \( \frac{2mn^2}{3n^2} \)  
   C. \( \frac{2mn^2}{3} \)  
   D. \( \frac{27}{32m^2} \)

23. **Travel Times** Use the formula \( \frac{d}{r} = t \) to find how long it takes to travel a distance of \( x^2 - 25 \) miles at the rate of \( x + 5 \) miles per hour.

24. **Error Analysis** Two students tried to find the axis of symmetry for the equation \( y = x^2 - 4x - 3 \). Which student is correct? Explain the error.

<table>
<thead>
<tr>
<th>Student A</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = \frac{-b}{2a} = \frac{-4}{2(1)} = -2 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Student B</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2 )</td>
</tr>
</tbody>
</table>

*25. **Multi-Step** The population of Ireland between the years 1901 and 2006 can be approximated by the function \( y = 0.000285x^2 - 0.0232x + 3.366 \), where \( x \) is the number of years after 1900 and \( y \) is the population for that year in millions of people.

   a. **Justify** Tell how you know the function has a minimum rather than a maximum.

   b. **Estimate** What was the minimum population and when did it occur? Round to the nearest year.

   c. **Predict** Use the function to predict the population of Ireland in 2020.

*26. **Geometry** The equation \( y = -x^2 + 35x \) gives the area of a rectangle with a fixed perimeter of 70 units, where \( x \) is the width of the rectangle. Explain how to use the equation to find the greatest possible area of the rectangle.

*27. **Multiple Choice** Simplify \( \frac{6}{9q^2} - \frac{3q}{9q^2} \).

   A. \( \frac{1}{3q} \)  
   B. \( \frac{2}{3q} \)  
   C. \( \frac{2 - q}{3q^2} \)  
   D. \( \frac{3(2 - q)}{q^2} \)

*28. **Business** Ms. Suarez owns a manufacturing company. Her total profits for one year are \( x^2 + 44x + 420 \), where \( x \) is the number of units manufactured. Her profits for the spring are \( x^2 + 7x + 100 \), while her profits for the winter are \( 17x + 40 \). Write a simplified expression for the fraction of the total profits that come from the spring and winter seasons.

*29. **Justify** When adding \( \frac{5n}{np} + \frac{6p}{np} \), why multiply the first expression by \( \frac{p}{p} \) and the second by \( \frac{n}{n} \)?

*30. **Write** Describe the steps in adding and simplifying \( \frac{2}{x^2 + 6x + 9} + \frac{x}{x + 3} \).
Choosing a Factoring Method

Whole numbers can be factored into prime numbers. For example, 30 can be written as a product of its factors: \(30 = 2 \cdot 3 \cdot 5\). A factorization can be verified as correct by multiplying the factors and verifying that the product is the same as the original whole number. Similarly, many polynomials, as shown below, can also be factored.

\[
\begin{align*}
x^3y + 2xy^2 & \quad x^3 + 7x^2 + 2x + 14 & \quad x^2 - 4 \\
x^2 + 16x + 64 & \quad x^2 + 8x + 15 & \quad 2x^3 + 7x + 3
\end{align*}
\]

The difference between factoring whole numbers and polynomials is that often the factors of polynomials are other polynomials and not whole numbers. Factorization can be checked through multiplication and simplification. If the original polynomial results after correctly multiplying and simplifying, the polynomial has been factored correctly. If not, try again, or the polynomial may in fact be prime.

Many possibilities and methods exist for factoring. To begin the factoring process, it is helpful to have a checklist.

**Checklist Item 1:** Look for the greatest common factor. Does each term have a common factor?

For example, factor \(x^3y + 2xy^2\).

1. Write each term of the polynomial as a product of its factors.
2. What does each term of the polynomial have in common?
3. Factor out the monomial from each term.
4. Now factor out the monomial from the polynomial.

**Checklist Item 2:** Look for a difference of two squares. Are there only two terms of the polynomial, and are they being subtracted? Are those two terms perfect squares?

For example, factor \(x^2 - 4\).

5. To begin the process of factoring, start with the first item on the checklist and proceed. Is there a common factor for all terms in the binomial?
6. Are there only two terms being subtracted?
7. The polynomial is a binomial of the form \(a^2 - b^2\). What is the value of \(a\) and \(b\)?
8. Use the factorization of the difference of two squares, \(a^2 - b^2 = (a + b)(a - b)\), to factor the binomial.
9. If the two terms of the binomial were perfect squares being added, \((x^2 + 4)\), could that binomial be factored similarly?

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**Checklist Item 3:** Look for perfect-square trinomials. Are the first and last terms perfect squares? Is the second term the product of the square roots of the first and last term?

For example, factor $x^2 + 16x + 64$.

10. Is there a common factor for all terms in the polynomial?
11. How many terms exist in the polynomial?
12. Are the first and last terms perfect squares? If so, what are their respective square roots?
13. Notice that the trinomial is of the form $a^2 + 2ab + b^2$. What is the value of $a$? $b$?
14. Use the formula $a^2 + 2ab + b^2 = (a + b)^2$ for factoring a perfect-square trinomial.
15. **Model** Is it possible to factor $x^2 + 8x + 32$ in a similar manner? Explain.

**Checklist Item 4:** Are there three terms of the polynomial, all of which are being added? Is the last term not a perfect square?

Some trinomials of the form $x^2 + bx + c$ that are not perfect-square trinomials can still be factored as the product of two binomials, such as $(x + j)(x + k)$, where $c = jk$ and $b = j + k$.

For example, factor $x^2 + 8x + 15$.

16. Is there a common factor for all terms in the polynomial?
17. Is this a perfect-square trinomial? Explain.
18. Is the trinomial of the form $x^2 + (j + k)x + jk$? If so, find two values that add to 8 and multiply to 15. What is the value of $j$? $k$?
19. Factor the polynomial using the formula for factoring a trinomial that is not a perfect square.

**Checklist Item 5:** Are there four terms in the polynomial? If you have four terms with no GCF, try to group the terms into smaller polynomials. Then factor each group by its GCF.

For example, factor $x^3 + 7x^2 + 2x + 14$.

20. Is there a common factor for all terms in the polynomial?
21. How many terms are in the polynomial?
22. Use parentheses to group the first two terms and the last two terms.
23. Factor out the GCF of the first group of terms.
24. Factor out the GCF of the second group of terms.
25. Write the polynomial using the factored groups of terms.
26. What factor does each term have in common?
27. Factor out the common term.
Some trinomials of the form $ax^2 + bx + c$ can also be factored. The following activity will demonstrate this factorization through the use of algebra tiles.

**Exploration**  
**Factoring Trinomials of the Form $ax^2 + bx + c$ Using Algebra Tiles**

Factor $2x^2 + 7x + 3$.

Collect two $x^2$-tiles, seven $x$-tiles, and three 1-tiles.

28. Build a rectangle using the algebra tiles. Keep the $x^2$-tiles on the top row of the rectangle. Fill in the $x$-tiles horizontally below and vertically to the right of the $x^2$-tiles. Use the 1-tiles to fill in the gaps of the rectangle. Remember that the lines between the tiles must be completely vertical or horizontal across the entire pattern.

Observe that the top edge of the pattern represents the length of a rectangle. The side edge of the pattern represents the width of a rectangle.

29. Using the sides without the $x^2$-tiles, how can the length and width of the rectangle be represented?

30. How is the area of a rectangle calculated?

31. Write an expression for the area of the rectangle using the length and width previously found.

32. What is the factored form of $2x^2 + 7x + 3$?

To review factoring polynomials, first look for common factors of the terms. If there is a common factor, find the GCF, and then factor it out of the polynomial. After factoring out the GCF, look for patterns.

**Pattern 1:** If the polynomial has two terms, then look for the difference of two squares.

**Pattern 2:** If the polynomial has three terms, look for a perfect-square trinomial. If the trinomial is not a perfect square, try to factor using the method for general trinomials of the form $x^2 + bx + c$ or $ax^2 + bx + c$.

**Pattern 3:** If the polynomial has four terms, try to factor by grouping.

Examine the resulting polynomial factor(s) to determine if any can be factored further. If so, continue the process. If not, factoring is complete. Remember that the factored result can always be verified through multiplication and simplification.
Factor the polynomial.

a. \( x^2 + 2x + 1 \)
b. \( 3x^2 + xy - 12x - 4y \)
c. \( 9y^4 - 1 \)

Identify the first step needed to factor the polynomial. Then use that method to factor the polynomial. List other methods used, if applicable.

d. \( 5x^4 - 5x^2 \)
e. \( 9x^2 + 30x + 25 \)
f. \( x^3 - 9 \)

Use algebra tiles to factor the following polynomials. Verify the factorization through multiplication and simplification.

g. \( 3x^2 + 13x + 4 \)
h. \( x^2 + 9x + 20 \)
i. \( 2x^2 + 8x + 6 \)